MATH 421/510
HW 3

Due: March 3, 2016 (in class)

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1. Chapter 4, Exercise 3.

2. (Two topologies on a Banach space) Let X be a Banach space. The topology induced by the norm is called the *strong topology* (More precisely, the topology induced by the metric associated to the norm). The *weak topology* on X is defined to be the weak topology generated by the dual space X^* (in the sense defined in page 120).

   (a) Show that every weakly closed set is strongly closed.

   (b) Show that if C is a convex and strongly closed, then C is weakly closed. (Hint: Use Hahn-Banach theorem)

   (c) The converse of (a) is not true in general. Consider the case $X = \ell^2$ and the unit sphere $S = \{x \in \ell^2 : \|x\| = 1\}$. Show that S is strongly closed but not weakly closed.

3. Chapter 4, Exercise 34.

4. (Metrizability of product topology) We say that a topology $(X, \mathcal{T})$ is *metrizable* if there is a metric d on X such that the topology induced by the metric d coincides with $\mathcal{T}$.

   Let (Y, d_Y) be a metric space and let A be an index set. Consider the space $X = Y^A$ equipped with product topology, where the space Y has the topology induced by the metric d_Y.

   (a) Show that $d_1 = \min(1, d_Y)$ is a metric on Y such that (Y, d_1) and (Y, d_Y) induce the same topology.

   (b) If A is countable, show that X is metrizable. (Hint: Use part (a) to construct a metric on X)

   (c) If A is uncountable and if Y has at least two points, show that X is not metrizable. (Hint: Show that the product topology on X is not first countable).