1. Let \((S, \mathcal{S}, P)\) be a probability space. Let \((S_i, \mathcal{S}_i)\) be measurable spaces for \(i = 1, 2\). Let \(X_i : (S, \mathcal{S}) \rightarrow (S_i, \mathcal{S}_i)\) be \(S_i\)-valued random variables for \(i = 1, 2\). Let \(\Phi : S_1 \times S_2 \rightarrow \mathbb{R}\) be a bounded measurable function with respect to the product \(\sigma\)-field \(\mathcal{S}_1 \times \mathcal{S}_2\).

(a) Show that for each \(x \in S_1\), the function \(\omega \mapsto \Phi(x, X_2(\omega))\) is a measurable \(\mathbb{R}\)-valued function on \((S, \mathcal{S})\).

(b) By part (a), for all \(x \in S_1\), we can define

\[
H(x) := \int_S \Phi(x, X_2(\omega)) \, dP(\omega).
\]

Show that the function \(x \mapsto H(x)\) is a measurable \(\mathbb{R}\)-valued function on \((S_1, \mathcal{S}_1)\).

(c) Let \(\mathcal{G}\) be a sub-\(\sigma\)-field of \(\mathcal{S}\) such that \(X_1\) is \(\mathcal{G}\)-measurable and \(X_2\) is independent of \(\mathcal{G}\). Then show that

\[
E(\Phi(X_1, X_2) | \mathcal{G})(\omega) = H(X_1(\omega))
\]

\(P\)-almost surely.

(Hint: Functional Montone Class Theorem for all the three parts)

2. (a) Let \((S, \mathcal{S})\) denote \((\mathbb{Z}_+, \mathcal{P}(\mathbb{Z}_+))\) and let \((\Omega, \mathcal{F}, P)\) be a probability space. Let \(\{Y_{n,k} : (\Omega, \mathcal{F}) \rightarrow (S, \mathcal{S}) : n \in \mathbb{Z}_+, k \in \mathbb{N}\}\) be a collection of iid copies of \(\mathbb{Z}_+\)-valued random variables, where \((S, \mathcal{S}) = (\mathbb{Z}_+, \mathcal{P}(\mathbb{Z}_+))\). For \(i \in S, A \in \mathcal{S}\) define \(p : S \times S \rightarrow [0, 1]\) by

\[
p(i, A) = P(\sum_{k=1}^{i} Y_{1,k} \in A).
\]

Show that \(p\) is a transition probability on \((S, \mathcal{S})\)
(b) Fix $x_0 \in \mathbb{Z}_+$ and define random variables $X_0 \equiv x_0$ and

$$X_{n+1} = \sum_{k=1}^{X_n} Y_{n,k}, \text{ for } n \in \mathbb{Z}_+.$$  

Let $\mathcal{F}_n = \sigma \{Y_{k,j} : k < n, j \in \mathbb{N}\}$. Show that $\{X_n : n \in \mathbb{Z}_+\}$ is a Markov chain with respect to $(\mathcal{F}_n)$ with transition probability $p$ and initial distribution $\delta_{x_0}$.

3. Consider a Markov chain $\{X_n : n \in \mathbb{Z}_+\}$ with general state space $(S, \mathcal{S})$ (not necessarily countable) defined on the cannonical product space $(\Omega, \mathcal{F}) = (S^{\mathbb{Z}_+}, \mathcal{S}^{\mathbb{Z}_+})$. If $f$ is a bounded measurable function on $S$, let

$$Gf(x) = E_x(f(X_1)) - f(x).$$

$G$ is called the generator of the Markov chain $\{X_n : n \in \mathbb{Z}_+\}$.

We say that $h$ is a harmonic function on the set $D \subset S$ iff $Gh(x) = 0$ for all $x \in D$.

(a) If $h$ is a bounded harmonic function on $S$ prove that $h(X_n)$ is an $(\mathcal{F}_n^X)$-martingale with respect to every $P_x$. If $A \in \mathcal{S}$ and $h$ is a bounded function on $S$ which is harmonic on $A^c$, show that $h(X_{n\land V_A})$ is an $(\mathcal{F}_n^X)$-martingale with respect to every $P_x$. Here $V_A := \inf \{n \geq 0 : X_n \in A\}$ is as defined in Exercise 5.2.11

(b) Let $A \in \mathcal{S}$ satisfy $P_x(V_A < \infty) = 1$ for every $x \in S$. Let $f : A \to \mathbb{R}$ be bounded and measurable. Prove that $h(x) = E_x(f(X_{V_A}))$ is the unique bounded function on $S$ which is harmonic on $A^c$ and equals $f$ on $A$.

(c) If $f : S \to \mathbb{R}$ is bounded and measurable, prove that $M^f_n = f(X_n) - \sum_{i=0}^{n-1} Gf(X_i)$ is an $(\mathcal{F}_n^X)$-martingale with respect to every $P_x$.

4. Exercise 5.2.6. ($S$ is countable. You may assume that $k \in \mathbb{N}$).

5. Exercise 5.2.11. ($S$ is countable. For (i) you need to verify that $g(x)$ is finite for all $x$. For (ii) you should verify the integrability of the martingale. For (iii) assume that the series in (*) is absolutely summable. For (iii) $g(x) = E_x \tau_A$ should read $g(x) = E_x V_A$ instead).
Practice Problems (do not hand in)

1. Let \( Z : (\Omega, \mathcal{F}) \to (S, \mathcal{S}) \) be a measurable map and let \( P \) be a probability measure on \((\Omega, \mathcal{F})\). Show that
\[
\nu(A) = P(Z^{-1}(A)) = P(\{\omega \in \Omega : Z(\omega) \in A\}), \quad \text{for all } A \in \mathcal{S},
\]
defines a probability measure on \((S, \mathcal{S})\). This measure \( \nu \) is called the 'pushforward measure of \( P \) under \( f \).

2. Let \( p : S \to S \to [0, 1] \) be a transition probability. Show that for all \( f \in \mathcal{bS} \), the map
\[
x \mapsto \int_S f(y) p(x, dy)
\]
is also in \( \mathcal{bS} \). (Hint: FMCT).

3. Let \( (\Omega, \mathcal{F}, P) \) be a probability space and let \( Y : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}) \) be a random variable. For all \( x \in \mathbb{R} \), defined \( p : \mathbb{R} \times \mathcal{B} \to [0, 1] \) be defined by
\[
p(x, A) = P(x + Y \in A).
\]
Let \( x_0 \in \mathbb{R} \). Let \( Y_j : \Omega \to \mathbb{R} \) be iid copies of \( Y \), for \( j \in \mathbb{N} \) and let \( \mathcal{F}_n = \mathcal{F}_n^Y = \sigma(Y_1, \ldots, Y_n) \). Let \( X_n = x_0 + \sum_{j=1}^{n} Y_j \). Show that \( p \) is a transition probability on \((\mathbb{R}, \mathcal{B})\) and that \( X_n \) is a Markov chain with respect to \( \mathcal{F}_n \) with transition probability \( p \).

4. Let \( \{X_n : n \in \mathbb{Z}_+\} \) be a Markov chain on \((\Omega, \mathcal{F}, P)\) with respect to the filtration \((\mathcal{F}_n)\). Show that the map \( X : (\Omega, \mathcal{F}) \to (S^{\mathbb{Z}_+}, \mathcal{S}^{\mathbb{Z}_+}) \) defined by
\[
\omega \mapsto (X_0(\omega), X_1(\omega), \ldots)
\]
is measurable. (Hint: \( \pi - \lambda \) theorem).
Remark: Practice problems 1. and 4. can be used to justify the probability measure on the canonical space (the measure \( P_X \) on \((S^{\mathbb{Z}_+}, \mathcal{S}^{\mathbb{Z}_+})\) given the initial distribution \( \mu \) and the transition probability \( p \) defined in class is unique). The measure \( P_X \) defined in class is simply the pushforward of \( P \) under the map \( X \).

5. Exercise 5.2.7.
6. Let \( \{X_n : n \in \mathbb{Z}_+\} \) be a stochastic process and let \((\mathcal{F}_n)_{n \in \mathbb{Z}_+}\) be a filtration on \((\Omega, \mathcal{F}, P)\). Let \(S, T : \Omega \to \mathbb{Z}_+ \cup \{\infty\}\) be \(\mathcal{F}_n\)-stopping times. Show the following:

(a) If \(X_\infty\) is \(\mathcal{F}\)-measurable and \(X_n\) is \((\mathcal{F}_n)\)-adapted, then \(\omega \mapsto X_{T(\omega)}(\omega)\) is \(\mathcal{F}_T\)-measurable.

(b) If \(S \leq T\) then \(\mathcal{F}_S \subseteq \mathcal{F}_T\).

(c) \(\mathcal{F}_{S \vee T} = \mathcal{F}_S \cap \mathcal{F}_T\).

(d) If \(F \in \mathcal{F}_{S \vee T}\) then \(F \cap \{S \leq T\} \in \mathcal{F}_T\).

(e) \(\mathcal{F}_{S \vee T} = \sigma(\mathcal{F}_S \cup \mathcal{F}_T)\).

7. Let \(X_i : (\Omega, \mathcal{F}) \to (S_i, \mathcal{S}_i)\) be measurable for \(i = 1, 2\). Show that the map \(\omega \mapsto (X_1(\omega), X_2(\omega))\) is a measurable map from \((\Omega, \mathcal{F})\) to the product space with product \(\sigma\)-field \((S_1 \times S_2, \mathcal{S}_1 \times \mathcal{S}_2)\) (we used this fact in the proof of Strong Markov property. Hint: \(\pi\)-\(\lambda\) theorem).