1. Let $\mu, \nu$ be measures on $(\Omega, \mathcal{F})$ such that $\mu$ is probability measure, and $\nu$ is a finite measure. We say that $\nu$ is uniformly absolutely continuous with respect to $\mu$, if for all $\epsilon > 0$ there exists $\delta > 0$ such that any $A \in \mathcal{F}$ with $\mu(A) < \delta$ satisfies $\nu(A) < \epsilon$.

Show that $\nu$ is uniformly absolutely continuous with respect to $\mu$ if and only if $\nu$ is absolutely continuous with respect to $\mu$.

2. Exercise 4.1.5

3. Exercise 4.1.6

4. Exercise 4.1.7

5. Exercise 4.1.8

6. Exercise 4.1.9

Practice problems (do not hand in)

1. Exercises 4.1.2, 4.1.3, 4.1.4.

2. Let $(\Omega, \mathcal{F})$ be a measurable space and let $X, Y : \Omega \to \mathbb{R}$ be random variables. Show that the following are equivalent:

   (a) $Y$ is $\sigma(X)$ measurable;

   (b) There exists a Borel function $h : \mathbb{R} \to \mathbb{R}$ such that $Y = h(X)$.

   (This is called the **Doob-Dynkin lemma**)

3. Let $X$ be an integrable random variable on $(\Omega, \mathcal{F})$. Let $\mathcal{G}$ denote the $\sigma$-field $\{\emptyset, \Omega\}$. Then show that the conditional expectation generalizes the notion expectation in the following sense:

   $$\mathbb{E}(X|\mathcal{G})(\omega) = \mathbb{E}X = \int_{\Omega} X \, dP,$$

   for all $\omega \in \Omega$, where $\mathbb{E}X$ is the expectation of $X$.

4. The purpose of this exercise is to understand some formulae for conditional probability in basic probability courses as special cases of our (measure theoretic) definition of conditional probability.
Let $B$ be an event in $(\Omega, \mathcal{F}, P)$ with $0 < P(B) < 1$. Prove that
\[
\mathbb{E}[1_A|\sigma(1_B)](\omega) = \frac{P(A \cap B)}{P(B)} 1_B(\omega) + \frac{P(A \cap B^c)}{P(B^c)} 1_{B^c}(\omega), \quad \text{for all } \omega \in \Omega.
\]

**Remark:** This explains the reason for the definition $P(A|B) = \frac{P(A \cap B)}{P(B)}$ and Bayes’ rule in basic probability courses.

Let $X,Y$ be random variables with joint probability density function (pdf) $f_{X,Y} : \mathbb{R} \times \mathbb{R} \to [0,\infty)$, that is
\[
P(X \leq x, Y \leq y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(s,t) \, ds \, dt, \quad \text{for all } x \in \mathbb{R}, y \in \mathbb{R}.
\]

Define the *conditional probability density function*
\[
f_{X|Y=y}(x) = \begin{cases} 
\frac{f_{X,Y}(x,y)}{\int_{\mathbb{R}} f_{X,Y}(s,y) \, ds} & \text{if } \int_{\mathbb{R}} f_{X,Y}(s,y) \, ds \neq 0, \\
0 & \text{if } \int_{\mathbb{R}} f_{X,Y}(s,y) \, ds = 0.
\end{cases}
\]

For all $y \in \mathbb{R}$, show that the function $f_{X|Y=y} : \mathbb{R} \to \mathbb{R}$ is Borel measurable. For any bounded Borel measurable function $g : \mathbb{R} \to \mathbb{R}$, show that
\[
\mathbb{E}[g(X)|\sigma(Y)] = h(Y),
\]
where $h : \mathbb{R} \to \mathbb{R}$ is the function
\[
h(y) = \int_{\mathbb{R}} f_{X|Y=y}(x) g(x) \, dx.
\]

Show that $\mathbb{E}[g(X)|\sigma(Y)] = h(Y)$ holds even if we assume $g$ to be a Borel measurable function (not necessarily bounded) such that $g(X)$ is integrable. (You should compare with Doob-Dynkin lemma)

**Instructions on submitting Homework:**

1. Solutions will be graded both on accuracy and quality of exposition. Solutions should be mathematically rigorous, well-crafted, and written in complete English sentences. Solutions must always be legible; use of LaTeX is encouraged and appreciated.

2. Please staple your pages together when you submit your assignment.