MATH 318 Assignment 7: Due Friday, Mar 9 at start of class

I. Problems to be handed in:

1. A certain coin comes up Heads with an unknown probability \( p \) each time it is tossed. It is tossed 6 times, giving 5 Heads.
   (a) Find an unbiased estimate for \( p \), giving your reasoning.
   (b) Assuming the coin is fair (\( p = 1/2 \)), what is the probability that the number of Heads is not in the range \( \{2, 3, 4\} \)?
   (c) Do the data provide grounds to reject the hypothesis that the coin is fair with 90% confidence? Hint: Use (b).

2. Ten measurements of the percentage of water in a methanol solution yielded the sample mean \( X = 0.552 \) and the sample variance \( S^2 = (0.037)^2 \). Assuming a normal distribution for the measurements, find a 95% confidence interval for the true percentage of water in the methanol solution.

3. We would like to test the null hypothesis \( \mu = 100 \), where \( \mu \) denotes the mean of a normally distributed random variable with variance \( \sigma^2 \). A sample of size 9 has sample mean \( \bar{X} = 105 \).
   (a) Find the \( p \)-value if the population standard deviation is known to be
      i. \( \sigma = 5 \)
      ii. \( \sigma = 10 \)
      iii. \( \sigma = 15 \)
   (b) In which of the three cases would the null hypothesis be rejected at the 5 percent level of significance?
   (c) In which of the three cases would the null hypothesis be rejected at the 1 percent level of significance?

4. (a) For the gamblers ruin problem, let \( M_i \) denote the expected number of games that will be played when Smith initially has \$i \ (i = 0, 1, \ldots, N) \). Let \( q = 1 - p \). Show that
   \[ M_0 = M_N = 0, M_i = 1 + pM_{i+1} + qM_{i-1} (i = 1, \ldots, N - 1) \]
   Hint: Compute the expectation of the number of games \( X \) by conditioning on the outcome of the first game:
   \[ EX = E[X|\text{win first game}]P(\text{win first game}) + E[X|\text{lose first game}]P(\text{lose first game}) \]
   conditional expectation is discussed in Chapter 3 and will be discussed soon in class.
   (b) Solve the equations in (a) to obtain
   \[ M_i = i(N - i) \text{ if } p = \frac{1}{2} \]
   \[ M_i = \frac{i}{q - p} - \frac{N}{q - p} \left(1 - \frac{q}{p}\right)^i \text{ if } p \neq \frac{1}{2} \]
   by proceeding as follows. First, find the general solution to the homogeneous equation \( M_i = pM_{i+1} + qM_{i-1} \) (already done in class). Next, find a particular solution to the inhomogeneous equation \( M_i = 1 + pM_{i+1} + qM_{i-1} \) (try \( M_i = ci \) for \( p = \frac{1}{2} \) and \( M_i = ci^2 \) for \( p \neq \frac{1}{2} \); find \( c \) that produces a solution). Add the general solution of the homogeneous equation to the particular
solution of the inhomogeneous equation. Finally, solve for the two unknown constants in the
general solution by using the boundary conditions.

5. This problem concerns 2-dimensional random walk on the square lattice \( \mathbb{Z}^2 \) (consisting of points \((x(1), x(2))\) with \(x(1)\) and \(x(2)\) both integers).

Let \( \vec{X} = (X(1), X(2)) \) be a random vector which takes the values \((1,0), (-1,0), (0,1), (0,-1)\) with equal probabilities 1/4. For \( j = 1, 2 \), the marginal p.m.f. of \( X^{(j)} \) takes value 1 with probability 1/4, 

\( -1 \) with probability 1/4, and 0 with probability 1/2.

Let \( \vec{X}_i \) be i.i.d. with the same distribution as \( \vec{X} \), and let \( \vec{S}_n = \vec{X}_1 + \ldots + \vec{X}_n \). Then \( \vec{S}_n \) represents the position after \( n \) steps of a random walker on \( \mathbb{Z}^2 \), starting from \((0,0)\), whose random steps are equally likely to be any of the four unit vectors.

(a) The expectation of a random vector \( \vec{Y} = (Y(1), Y(2)) \) is given by \( E\vec{Y} = (EY(1), EY(2)) \) and

the variance is given by \( \text{Var}(Y) = E[(Y - EY) \cdot (Y - EY)] \) with the dot indicating the dot product. Show that, for the random walk, \( E\vec{S}_n = (0,0) \) and \( \text{Var}(\vec{S}_n) = n \).

(b) If the random walk is instead 1-dimensional (probability 1/2 of steps left or right) or 3-
dimensional (probability 1/6 of steps north, south, east, west, up, down), what is the expected
position and the variance of the walk after \( n \) steps?

II. Recommended problems:

A. Let \( \Gamma(n) = \int_0^\infty e^{-t}t^{n-1} \, dt \). Prove that \( \Gamma(n) = (n-1)! \) if \( n = 1, 2, \ldots \) by two methods (i) use integration by parts to show that \( \Gamma(n) = (n-1)\Gamma(n-1) \), (ii) consider \( \frac{d^n}{dx^n} \int_0^\infty e^{-\lambda t} \, dt \) and set \( \lambda = 1. \)

B. Show that \( \Gamma(1/2) = \sqrt{\pi} \) by substituting \( t = s^2/2 \) in A.

C. 25 measurements are made of the splitting tensile stress (lb/in\(^2\)) of concrete cylinders. The

following table shows the frequency of each measured value, with the strength on the
rst line and the frequency on the second line. Assuming a normal distribution, determine a 99%
confidence interval for the mean splitting tensile stress \( \mu \) of the population from which the sample
was drawn. \([350.3, 369.7]\).

\[
\begin{array}{cccccccc}
320 & 330 & 340 & 350 & 360 & 370 & 380 & 390 \\
1 & 1 & 3 & 3 & 8 & 3 & 5 & 1 \\
\end{array}
\]

D. Chapter 4 \#57 \([p(1 - q/p)/(1 - (q/p)^n)] + q(1 - p/q)/(1 - (p/q)^n)\], \#58

E. Consider the gambler’s ruin scenario discussed in class, in which Smith has initial fortune \( n \) and
the bank has initial fortune \$(1000 - n)\), so the total is \$1000. Smith plays roulette and bets on
red. Determine the value of \( n \) so that the probability is 99% that Smith goes broke. \([956]\)