I. Problems to be handed in:

1. Let \( \{N_t : t \geq 0\} \) be a Poisson process of rate \( \lambda \), and let \( S_n \) denote the time of the \( n \)-th event. Find
   (a) \( E(N_5) \)
   (b) \( E(S_3) \)
   (c) \( P(N_5 < 3) \)
   (d) \( P(S_3 > 5) \)
   (e) \( P(S_3 > 5 | N_2 = 1) \)

2. A pizzeria sells meat and vegetarian pizzas. The number of meat orders by time \( t \) is a Poisson process \( (M_t)_{t \geq 0} \) with rate \( \mu \). The number of vegetarian orders is a Poisson process \( (V_t)_{t \geq 0} \) with rate \( \nu \). These two Poisson processes are assumed to be independent. It is a theorem that the sum of two independent Poisson processes with rates \( \mu \) and \( \nu \) is itself a Poisson process whose rate is \( \mu + \nu \).
   (a) What is the distribution of the total number of orders by time \( t \)?
   (b) Given that \( n \) orders are received by time \( t \), find and identify the conditional distribution of the number of meat orders that have arrived by time \( t \).

3. A (different) pizzeria sells meat and vegetarian pizzas. Unlike the previous pizzeria, the customers have no choice in the kind of pizza they receive. Each customer either receives a meat pizza with probability \( p \), or a vegetarian pizza with probability \( 1 - p \). In other words, the pizzeria chooses the kind of pizza that the customers receive. Assume that this choice made by the pizzeria is independent for each customer. The number of customers by time \( t \) is a Poisson process \( (N_t)_{t \geq 0} \) with rate \( \lambda \). Let \( M_t \) denote the number of customers who receive a meat pizza by time \( t \). Similarly, let \( V_t = N_t - M_t \) denote the number of customers who receive a vegetarian pizza by time \( t \).
   (a) What is the distribution of \( M_t \)? What is the distribution of \( V_t \)?
   (b) Are \( M_t \) and \( V_t \) independent for each \( t \)?

4. (a) Suppose that \( X_1, X_2, \ldots \) are independent Gaussian random variables, with \( X_i \sim N(\mu_i, \sigma_i^2) \). Let \( S_n = X_1 + X_2 + \ldots + X_n \). Compute the characteristic function of \( S_n \) and thereby identify its distribution.
   (b) Four fish are caught in a day. Their weights (in pounds) are independent \( N(5, 4) \) random variables (this is an approximation – in reality the weights cannot be negative). Find the probability that the last fish weighs more than the other three together.
   Hint: Consider \( X_1 + X_2 + X_3 - X_4 \); if \( X_4 \sim N(5, 4) \) then what is the distribution of \( -X_4 \).
   (c) Now assume in (a) that \( \mu_i = \mu \) and \( \sigma_i = \sigma \) for all \( i \). Let \( Y_n = n^{-1} S_n \) denote the average of the first \( n \) \( X_i \)'s. Identify the distribution of \( Y_n \) by calculating its characteristic function. Do the same for \( Z_n = n^{-1/2} \sum_{i=1}^n X_i \).
   (d) Show explicitly that the limit, as \( n \to \infty \), of the characteristic function of \( Y_n \) approaches the characteristic function of a constant random variable (as in the proof of the weak law of large numbers).
(e) Suppose $\mu = 0$ and $\sigma = 1$, so that each $X_i$ is a standard normal random variable. Compare the probabilities that $|Y_n| \leq 0.1$, for $n = 1, n = 5, n = 50$, and $n = 500$.

5. The standard Cauchy random variable has probability density function $f(x) = \frac{1}{\pi(1+x^2)}$ and characteristic function $\phi(t) = e^{-|t|}$. Suppose that $X_1, X_2, \ldots$ are independent standard Cauchy random variables and let $S_n = \sum_{i=1}^{n} X_i$.
   (a) We have seen that $EX_1$ is undefined. Check this via the characteristic function.
   (b) Use characteristic function to show that $n^{-1}S_n$ is also a standard Cauchy random variable.
      (This helps explain what you observed in Assignment 4 #6(b).)
   (c) Why does (b) not contradict the weak law of large numbers?

6. Suppose that the number of decades between occurrence of two serious earthquakes in a region follows an exponential distribution with parameter 1. In parts (b,c), print an submit your Octave scripts and plots.
   (a) Let $Y$ denote the number of earthquakes in a period of 100 decades. What is the distribution of $Y$?
   (b) Use Octave, generate independent random variables $X_1, X_2, \ldots, X_{10000}$, each of which gives the total number of earthquakes that occur in a simulation of a period of 100 decades. Present the results in a histogram which shows the number of 100 decade periods (among 10,000) which produced any given total number of earthquakes. (Given the vector $X$, use the command `hist(X,[50,150])`)
   (c) Using the same simulation as in (b), for $1 \leq i \leq 10000$, let $M_i$ denote the number of $j \in \{1, \ldots, i\}$ such that $X_j = 100$. Plot a graph of $M_i/i$ vs $i$, for $1 \leq i \leq 10000$.
   (d) In (c), what is the limiting value and why?
      (You may find it useful to recall Stirling’s approximation $n! \approx \sqrt{2\pi n} n^{n+\frac{1}{2}} e^{-n}$.)

II. Recommended problems: These provide additional practice but are not to be handed in.
A. Let $X,Y$ be independent exponential random variables with parameters $\lambda, \mu$ respectively. Show that the conditional distribution of $X$, given that $X < Y$ is $\text{Exp}(\lambda + \mu)$.
B. Let $X$ be distributed geometrically with parameter $p$. Compute the characteristic function of $X$ and use it to show that the variance of $X$ is $(1-p)/p^2$.
C. Chapter 2: #3, 57*.
D. Chapter 5: #54 (1, p, 2p(1, 1) − 1).
E. In problem 3, are $M_t$ and $V_s$ independent for each $t, s > 0$?