MATH 318 Assignment 1: Due Friday, Jan 12 at start of class

I. Problems to be handed in: The solutions should have clear explanations, not merely answers. In particular, for problems involving permutations and combinations, be sure to explain all factors arising in your solution.

1. Ross Chapter 1 #4
2. A coin is tossed until either there are more heads than tails, or until the fifth toss, whichever comes first. Write down the sample space, and determine the probability of each outcome in the sample space.
3. An ecology graduate student goes to a pond and captures 40 water beetles, marks each with a dot of paint and releases them. A few days later she goes back, captures another sample of 60, finding 12 marked beetles and 48 unmarked.
   (a) Assuming that the pond contains n beetles, determine the probability $L(n)$ that a catch of 60 beetles contain 12 marked ones.
   (b) Show that $L(n)$ is initially an increasing function of n which then becomes decreasing after reaching a maximum value. Find the *maximum likelihood estimate* for n: that is the value of n which maximizes $L(n)$.
   Hint: When does $L(n) \le L(n-1)$ hold?
4. A coin is tossed $2n$ times.
   (a) Compute the probability $P(n)$ that exactly half the outcomes are heads.
   (b) Calculate $P(n+1)/P(n)$ and conclude that $P(n)$ decreases as n increases, *i.e.,* $P(n+1) < P(n)$.
   (c) We will show that $P(n)$ goes to 0 as $n \to \infty$ like the function $\frac{b}{\sqrt{n}}$, where $b > 0$. Using *Stirling's formula*
   $$\lim_{n\to\infty} \frac{\sqrt{(2\pi)}n^{n+\frac{1}{2}}e^{-n}}{n!} = 1,$$
   show that
   $$\lim_{n\to\infty} \frac{b}{\sqrt{n}} \frac{1}{P(n)} = 1,$$
   for a suitably chosen b. What is the value of b?
   Hint: If $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, then $\lim_{n\to\infty} a_n b_n = AB$.
5. A coin is tossed $3n$ times.
   (a) Compute the probability $Q(n)$ that exactly a third of the outcomes are heads.
   (b) We will show that $Q(n)$ goes to 0 as $n \to \infty$ like the function $f(n) = \frac{c}{\sqrt{n}}e^{-an}$, where $a, c > 0$. Show that the limit
   $$\lim_{n\to\infty} \frac{c}{\sqrt{n}} e^{-an} \frac{1}{Q(n)} = 1,$$
   for suitably chosen values of $a, c > 0$. Compute a, c.
   Hint: Use Stirling's formula, and $x^n = e^{n \log x}$.

   Remark: Please compare 4(c) with 5(b) and observe that $Q(n)$ is *much smaller* than $P(n)$ for large values of n. *Large deviations theory* deals with estimating such exponentially small probabilities.

6. Use Octave (or MATLAB) to write a program that will do the following.
   (a) Write a function `birthday(n)` that:
      (i) generated a vector containing n numbers uniformly distributed on $\{1, 2, \ldots, 365\}$ (think of this as the list of birthdays of n people, use the command `unidrnd(365,1,n)`,
      (ii) returns 1 if there is at least one pair of people with coinciding birthdays (a "match") and 0 otherwise.
   (b) For $n = 2$ to 60, run the function `birthday(n)` 1000 times, and compute the proportion $X(n)$ of the 1000 times in which there was a match. Hint: set `A(n,i)=birthday(n)` for $i = 1, \ldots, 1000$ and put $X(n) = \frac{1}{1000} \sum_{i=1}^{n} A(n, i)$.
   (c) Let
$$Y(n) = 1 - \frac{365 \cdot 364 \cdots (365 - n + 1)}{365^n}.$$
   On a single graph, plot $X(n)$ and $Y(n)$ vs $n \in [2, 60]$.

II. Recommended problems: These provide additional practice but are not to be handed in. Starred problems have solutions in the text, and answers are given otherwise.
Ross, Chapter 1: 2*, 5*, 6, 7, 8, 9*, 10, 11 ($\frac{1}{36}, \frac{2}{36}, \frac{3}{36}, \frac{4}{36}, \frac{5}{36}, \frac{6}{36}, \frac{5}{36}, \frac{4}{36}, \frac{3}{36}, \frac{2}{36}, \frac{1}{36}$), 17*, 20 ($\frac{5}{12}$).