DAVIES’ METHOD FOR ANOMALOUS DIFFUSIONS

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ABSTRACT. Davies’ method of perturbed semigroups is a classical technique to obtain off-diagonal upper bounds on the heat kernel. However Davies’ method does not apply to anomalous diffusions due to the singularity of energy measures. In this note, we overcome the difficulty by modifying the Davies’ perturbation method to obtain sub-Gaussian upper bounds on the heat kernel. Our computations closely follow the seminal work of Carlen, Kusuoka and Stroock [8]. However, a cutoff Sobolev inequality due to [1] is used to bound the energy measure.

1. Introduction

Davies’ method of perturbed semigroups is a well-known method to obtain off-diagonal upper bounds on the heat kernel. It was introduced by E. B. Davies to obtain the explicit constants in the exponential term for Gaussian upper bounds [9] using the logarithmic-Sobolev inequality. Davies’ method was extended by Carlen, Kusuoka and Stroock to a non-local setting [8, Section 3] using Nash inequality. Moreover, Davies extended this technique to higher order elliptic operators on $\mathbb{R}^n$ [10, Section 6 and 7]. More recently Barlow, Grigor’yan and Kumagai applied Davies’ method as presented in [8] to obtain off-diagonal upper bounds for the heat kernel of heavy tailed jump processes [6, Section 3].

Despite these triumphs, Davies’ perturbation method has not yet been made to work in the following contexts:

(a) Anomalous diffusions (See [3, Section 4.2]).
(b) Jump processes with jump index greater than or equal to 2 (See [15, Remark 1(d)] and [11, Section 1]).

The goal of this work is to extend Davies’ method to anomalous diffusions in order to obtain sub-Gaussian upper bounds. In the anomalous diffusion setting, we use cutoff functions satisfying a cutoff Sobolev inequality to perturb the corresponding heat semigroup. We use a recent work of Andres and Barlow [1] to construct these cutoff functions. We will extend the techniques developed here in a sequel to a non-local setting for the jump processes mentioned in (b) above [16].

Before we proceed, we briefly outline Davies’ method as presented in [8] and point out the main difficulty in extending it to the anomalous diffusion setting.

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Consider a metric measure space \((M, d, \mu)\) and a Markov semigroup \((P_t)_{t \geq 0}\) symmetric with respect to \(\mu\). Instead of considering the original Markov semigroup \((P_t)_{t \geq 0}\), we consider the perturbed semigroup

\[
(P^\psi_t f)(x) = e^{\psi(x)} \left( P_t \left( e^{-\psi} f \right) \right)(x)
\]

where \(\psi\) is a ‘sufficently nice function’. Given an ultracontractive estimate

\[
\|P_t\|_{1 \to \infty} \leq m(t)
\]

for the diffusion semigroup, Davies’ method yields an ultracontractive estimate for the perturbed semigroup

\[
\|P^\psi_t\|_{1 \to \infty} \leq m_\psi(t).
\]

If \(p_t(x, y)\) is the kernel of \(P_t\), then the kernel of \(P^\psi_t\) is

\[
p^\psi_t(x, y) = e^{-\psi(x)} p_t(x, y) e^{\psi(y)}.
\]

Therefore by (3), we obtain the off-diagonal estimate

\[
p_t(x, y) \leq m_\psi(t) \exp(\psi(y) - \psi(x)).
\]

By varying \(\psi\) over a class of ‘nice functions’ to minimize the right hand side of (4), Davies obtained off-diagonal upper bounds. In Davies’ method as presented in \([9, 8]\), it is crucial that the function \(\psi\) satisfies

\[
e^{-2\psi} \Gamma(e^\psi, e^\psi) \ll \mu \quad \text{and} \quad e^{2\psi} \Gamma(e^{-\psi}, e^{-\psi}) \ll \mu,
\]

where \(\Gamma(\cdot, \cdot)\) denotes the corresponding energy measure. In fact the expression of \(m_\psi\) in (3) depends on the uniform bound on the Radon-Nikodym derivatives of the energy measure given by (See \([8, \text{Theorem 3.25}]\))

\[
\Gamma(\psi) := \left\| \frac{d e^{-2\psi} \Gamma(e^\psi, e^\psi)}{d\mu} \right\|_\infty \vee \left\| \frac{d e^{2\psi} \Gamma(e^{-\psi}, e^{-\psi})}{d\mu} \right\|_\infty.
\]

The main difficulty in extending Davies’ method to anomalous diffusions is that, for many ‘typical fractals’ that satisfy a sub-Gaussian estimate, the energy measure \(\Gamma(\cdot, \cdot)\) is singular with respect to the underlying symmetric measure \(\mu\) \([12, 14, 7]\). This difficulty is well-known to experts (for instance, \([4, \text{p. 1507}]\) or \([13, \text{p. 86}]\)). In this context, the condition \(e^{-2\psi} \Gamma(e^\psi, e^\psi) \ll \mu\) implies that \(\psi\) is necessarily a constant, in which case the off-diagonal estimate of (4) is not an improvement over the diagonal estimate (3).

Let \((M, d, \mu)\) be a locally compact metric measure space where \(\mu\) is a positive Radon measure on \(M\) with \(\text{supp}(\mu) = M\). We denote by \((\cdot, \cdot)\) the inner product on \(L^2(M, \mu)\). We consider a regular strongly local Dirichlet form \((\mathcal{E}, \mathcal{F})\) with generator \(-\mathcal{L}\), where \(\mathcal{L}\) is a positive definite, self-adjoint operator. That is

\[
\mathcal{E}(f, g) = -\langle \mathcal{L} f, g \rangle \text{ for all } f \in \mathcal{D}(\mathcal{L}), f \in \mathcal{F}.
\]

Let \((P_t)_{t \geq 0}\) denote the associated semigroup and let \(p_t(\cdot, \cdot)\) be the (regularised) kernel of \(P_t\) with respect to \(\mu\) \([1, \text{eq. (1.10)}]\). We denote by \(B(x, r) := \{y \in M : d(x, y) < r\}\) the ball centered at \(x\) with radius \(r\) and by

\[
V(x, r) := \mu(B(x, r))
\]
the corresponding volume. We assume that the metric measure space is Ahlfors-
regular: meaning that there exist \( C_1 > 0 \) and \( d_f > 0 \) such that

\[
C_1^{-1} r^{d_f} \leq V(x, r) \leq C_1 r^{d_f}
\]

for all \( x \in M \) and for all \( r \geq 0 \). The quantity \( d_f > 0 \) is called the \textit{volume growth exponent} or \textit{fractal dimension}. We are interested in obtaining sub-Gaussian upper bounds of the form

\[
p_t(x, y) \leq \frac{C_1}{t^{d_f/d_w}} \exp \left( -C_2 \left( \frac{d(x, y)^{d_w}}{t} \right)^{1/(d_w-1)} \right) \quad \text{USG}(d_f, d_w)
\]

where \( d_w \geq 2 \) is the \textit{escape time exponent} or \textit{walk dimension}. Such sub-Gaussian estimates are typical of many fractals. We assume the on-diagonal bound corresponding to the sub-Gaussian estimate of \( \text{USG}(d_f, d_w) \). That is, we assume that there exists \( C_1 > 0 \) such that

\[
p_t(x, x) \leq \frac{C_1}{t^{d_f/d_w}} \quad (5)
\]

for all \( x \in M \) and for all \( t > 0 \). The on-diagonal estimate of (5) is equivalent to the following Nash inequality ([8, Theorem 2.1]): there exists \( C_N > 0 \) such that

\[
\|f\|_2 \frac{2^{(d_w/d_f)} - 1}{d_w/d_f} \leq C_N \mathcal{E}(f, f)^{1/2} \|f\|_1^{2/d_w/d_f} \quad (6)
\]

\[
\mathcal{E}(f, g) = \int_M \Gamma(f, g)
\]

where the energy measure \( \Gamma \) is defined as follows. For any essentially bounded \( f \in \mathcal{F} \), \( \Gamma(f, f) \) is the unique Borel measure on \( M \) (called the energy measure) on \( M \) satisfying

\[
\int_M g d\Gamma(f, f) = \mathcal{E}(f, fg) - \frac{1}{2} \mathcal{E}(f^2, g)
\]

for all essentially bounded \( g \in \mathcal{F} \); \( \Gamma(f, g) \) is then defined by polarization. We shall use the following properties of the energy measure.

(i) \textit{Locality}: For all functions \( f, g \in \mathcal{F} \) and all measurable sets \( G \subset M \) on which \( f \) is constant

\[
1_G d\Gamma(f, g) = 0
\]

(ii) \textit{Leibniz and chain rules}: For \( f, g \in \mathcal{F} \) essentially bounded and \( \phi \in C^1(\mathbb{R}) \),

\[
d\Gamma(fg, h) = fd\Gamma(g, h) + gd\Gamma(f, h) \quad (6)
\]

\[
f\Gamma(\phi(f), g) = \phi'(f) d\Gamma(f, g). \quad (7)
\]
We wish to obtain an off-diagonal estimate using Davies’ perturbation method. The main difference from the previous implementations of the method is that, in addition to an on-diagonal upper bound (or equivalently Nash inequality), we also require a cutoff Sobolev inequality. Spaces satisfying the sub-Gaussian upper bound given in $\text{USG}(d, d_w)$ necessarily satisfy the cutoff Sobolev annulus inequality $\text{CSA}(d_w)$, a condition introduced by Andres and Barlow [1]. The condition CSA simplifies the cut-off Sobolev inequalities CS which were originally introduced by Barlow and Bass [4] for weighted graphs. The significance of the cut-off Sobolev inequalities CS and CSA is that they are stable under bounded perturbations of the Dirichlet form (Cf. [1, Corollary 5.2]). Moreover, the condition CS is stable under quasi-isometries of the underlying space [5, Theorem 2.21(b)]. Therefore cutoff Sobolev inequalities provide a robust method to obtain heat kernel estimates with anomalous time-space scaling.

**Definition 1.1.** Let $U \subset V$ be open sets in $M$ with $U \subset \bar{U} \subset V$. We say that a continuous function $\phi$ is a cutoff function for $U \subset V$ if $\phi \equiv 1$ on $U$ and $\phi \equiv 0$ on $V^c$.

**Definition 1.2.** ([1, Definition 1.10]) We say $\text{CSA}(d_w)$ holds if there exists $C_1, C_2 > 0$ such that for every $x \in M$, $R > 0$, $r > 0$, there exists a cutoff function $\phi$ for $B(x, R) \subset B(x, R + r)$ such that if $f \in F$, then

$$
\int_U f^2 \, d\Gamma(\phi, \phi) \leq C_1 \int_U \phi^2 \, d\Gamma(f, f) + \frac{C_2}{r^{d_w}} \int_U f^2 \, d\mu, \quad \text{CSA}(d_w)
$$

where $U = B(x, R + r) \setminus \bar{B}(x, r)$.

It is clear that the condition $\text{CSA}(d_w)$ is preserved by bounded perturbations of the Dirichlet form. The above definition is slightly different to the one introduced in [1, Definition 1.10]. However both definitions are equivalent due to a ‘self-improving’ property of $\text{CSA}(d_w)$ [1, Lemma 5.1].

Our main result is that the Nash inequality $\text{N}(d_f, d_w)$ and the cutoff Sobolev inequality $\text{CSA}(d_w)$ imply the desired sub-Gaussian estimate $\text{USG}(d_f, d_w)$. More precisely,

**Theorem 1.3.** Let $(M,d,\mu)$ be a locally compact metric measure space that satisfies $V(d_f)$ with volume growth exponent $d_f$. Let $(\mathcal{E}, \mathcal{F})$ be a strongly local, regular, Dirichlet form whose energy measure $\Gamma$ satisfies the cutoff Sobolev inequality $\text{CSA}(d_w)$ for some $d_w > 2$. Then the Nash inequality $\text{N}(d_f, d_w)$ implies the sub-Gaussian upper bound $\text{USG}(d_f, d_w)$.

**Remark.** The above properties given by $V(d_f)$ and $\text{USG}(d_f, d_w)$ are a special case of the more general assumptions of volume doubling and heat kernel upper bounds with a general time-space scaling of [1]. In fact, Theorem 1.3 is subsumed by [1, Theorem 1.12]. However, our methods give an alternate proof to [1, Theorem 1.12] in a restricted setting. Moreover, the techniques developed here will lead to new results which are not obtained by other methods. In particular, we will extend
these techniques to a non-local setting in a sequel [16] and resolve the conjecture posed in [15, Remark 1(d)].

2. Off diagonal estimates using Davies’ method

Spaces satisfying have a rich class of cutoff functions with low energy. We start by studying energy estimates of these cutoff functions.

2.1. Self-improving property of CSA. The cutoff Sobolev inequality CSA($d_w$) has a self-improving property which states that the constants $C_1, C_2$ in CSA($d_w$) are flexible. For example, we can decrease the value of $C_1$ in CSA($d_w$) by increasing $C_2$ appropriately. This is quantified in Lemma 2.1. Lemma 2.1 is essentially contained in [1]; we simplify the proof and extract from the same methods.

Lemma 2.1. Let $(M, d, \mu)$ satisfy $V(d f)$. Let $(\mathcal{E}, \mathcal{F})$ denote a strongly local, regular, Dirichlet form with energy measure $\Gamma$ that satisfies CSA($d_w$). There exists $C > 0$ such that for each $\rho \in (0, 1]$, there exists a cutoff function $\phi_\rho$ for $B(x, R) \subset B(x, R + r)$ that satisfies

$$\int_U f^2 \, d\Gamma(\phi_\rho, \phi_\rho) \leq 4C_1 \rho^2 \int_U d\Gamma(f, f) + \frac{C_2 \rho^{2-d_w}r}{r d_w} \int_U f^2 \, d\mu$$

(8)

for all $f \in \mathcal{F}$, where $C_1, C_2$ are the constants in CSA($d_w$). Further the cutoff function $\phi_\rho$ above satisfies

$$\left| \phi_\rho(y) - \left( \frac{R + r - d(x, y)}{r} \right) \right| \leq 2\rho$$

(9)

for all $y \in B(x, R + r) \setminus B(x, R)$.

Proof. Let $x \in M$, $r > 0$, $R > 0$, $\rho > 0$. Define $n := \lfloor \rho^{-1} \rfloor \in [\rho^{-1}/2, \rho^{-1}]$. We divide the annulus $U = B(x, R + r) \setminus B(x, R)$ into $n$-annuli $U_1, U_2, \ldots, U_n$ of equal width, where

$$U_i := B(x, R + ir/n) \setminus B(x, R + (i - 1)r/n), \quad i = 1, 2, \ldots, n.$$

By CSA($d_w$), there exists a cutoff function $\phi_i$ for $B(x, R + (i - 1)r/\rho) \subset B(x, R + ir/n)$ satisfying

$$\int_{U_i} f^2 \, d\Gamma(\phi_i, \phi_i) \leq C_1 \int_{U_i} d\Gamma(f, f) + \frac{C_2}{(r/n)d_w} \int_{U_i} f^2 \, d\mu$$

(10)

for $i = 1, 2, \ldots, n$. We define $\phi = n^{-1} \sum_{i=1}^n \phi_i$. By locality, we have

$$d\Gamma(\phi, \phi) = \frac{1}{n^2} \sum_{i=1}^n d\Gamma(\phi_i, \phi_i).$$

(11)
Therefore by \((11), (10)\) and \(\rho^{-1}/2 \leq n = |\rho^{-1}| \leq \rho^{-1}\), we obtain
\[
\int_U f^2 \, d\Gamma(\phi, \phi) = n^{-2} \sum_{i=1}^n \int_{U_i} f^2 \, d\Gamma(\phi, \phi)
\leq n^{-2} \sum_{i=1}^n \left( C_1 \int_{U_i} d\Gamma(f, f) + \frac{C_2}{(r/n)^d_w} \int_{U_i} f^2 \, d\mu \right)
\leq C_1 n^{-2} \int_U d\Gamma(f, f) + \frac{C_2 n^{d_w-2}}{r^{d_w}} \int_{U_i} f^2 \, d\mu
\leq 4C_1 \rho^2 \int_U d\Gamma(f, f) + \frac{C_2 \rho^2-d_w}{r^{d_w}} \int_{U_i} f^2 \, d\mu.
\]
This completes the proof of \((8)\).

Note that if \(y \in U_i\), then \(1 - i/n \leq \phi(y) \leq 1 - (i-1)/n\) and \(R + (i-1)r/n \leq d(x, y) < R + ir/n\), for each \(1 \leq i \leq n\). This along with \(n^{-1} \leq 2\rho\) implies \((9)\).

Observe that by \((9)\), the cutoff function \(\phi_\rho\) for \(B(x, r) \subset B(x, R + r)\) satisfies
\[
\lim_{\rho \downarrow 0} \phi_\rho(y) = 1 \wedge \left( \frac{(R + r - d(x, y))_+}{r} \right).
\]

2.2. Estimates on perturbed forms. The key to carry out Davies’ method is the following elementary inequality.

**Lemma 2.2.** Let \((E, F)\) be a strongly local, regular, Dirichlet form. Then
\[
E(e^\psi f^{2p-1}, e^{-\psi} f^p) \geq \frac{1}{p} E(f^p, f^p) - p \int_M f^{2p} \, d\Gamma(\psi, \psi)
\] \hspace{1cm} (12)
for all \(f \in F, \psi \in F\) and \(p \in [1, \infty)\).

**Proof.** Using Leibniz rule \((6)\) and chain rule \((7)\), we obtain
\[
\Gamma(e^\psi f^{2p-1}, e^{-\psi} f^p) - \frac{1}{p} \Gamma(f^p, f^p) - p f^{2p} \Gamma(\psi, \psi)
= (p - 1) \left( f^{2(p-1)} \Gamma(f, f) + f^{2p} \Gamma(\psi, \psi) - 2 f^{2p-1} \Gamma(f, \psi) \right).
\] \hspace{1cm} (13)
By [8, Theorem 3.7] and Cauchy-Schwarz inequality, we have
\[
\int_M f^{2p-1} \, d\Gamma(f, \psi) \leq \sqrt{\int_M f^{2(p-1)} \, d\Gamma(f, f) \cdot \int_M f^{2p} \, d\Gamma(\psi, \psi)}. \]
Therefore
\[
2 \int_M f^{2p-1} \, d\Gamma(f, \psi) \leq \int_M f^{2(p-1)} \, d\Gamma(f, f) + \int_M f^{2p} \, d\Gamma(\psi, \psi). \] \hspace{1cm} (14)
By integrating \((13)\) and using \((14)\), we obtain \((12)\). □
Davies used the bound
\[
\int_M f^{2p} d\Gamma(\psi, \psi) \leq \left\| \frac{d\Gamma(\psi, \psi)}{d\mu} \right\|_\infty \|f\|_{2p}^2
\]
to control a term in (12). However for anomalous diffusions, the energy measure is singular to \(\mu\). We will instead use CSA\((d_w)\) to bound \(\int_M f^{2p} d\Gamma(\psi, \psi)\) by choosing \(\psi\) to be a multiple of the cutoff function satisfying CSA\((d_w)\). The following estimate is analogous to [8, Theorem 3.9] but unlike in [8], the cutoff functions depend on both \(p\) and \(\lambda\). This raises new difficulties in the implementation of Davies’ method.

**Proposition 2.3.** Let \((M, d, \mu)\) be a metric measure space. Let \((E, F)\) be a strongly local, regular, Dirichlet form on \(M\) satisfying CSA\((d_w)\). There exists \(C > 0\) such that, for all \(\lambda \geq 1\), for all \(r > 0\), for all \(x \in M\) and for all \(p \in [1, \infty)\), there exists a cutoff function \(\phi = \phi_{p, \lambda}\) on \(B(x, r) \subset B(x, 2r)\) such that
\[
\mathcal{E}(e^{\lambda \phi} f^{2p-1}, e^{-\lambda \phi} f) \geq \frac{1}{2p} \mathcal{E}(f^p, f^p) - C \frac{\lambda^{d_w} d_w}{r^{d_w}} \|f\|_{2p}^2.
\]
for all \(f \in F\). There exists \(C' > 0\) such that the cutoff functions \(\phi_{p, \lambda}\) above satisfy
\[
\|\exp (\lambda (\phi_{p, \lambda} - \phi_{2p, \lambda}))\|_\infty \lor \|\exp (-\lambda (\phi_{p, \lambda} - \phi_{2p, \lambda}))\|_\infty \leq \exp (C'/p)
\]
for all \(p \geq 1\) and for all \(\lambda \geq 1\).

**Proof.** This Theorem follows from Lemma 2.2 and Lemma 2.1. Let \(x \in M\) and \(r > 0\) be arbitrary. Using (12), we obtain
\[
\mathcal{E}(e^{\lambda \phi} f^{2p-1}, e^{-\lambda \phi} f) \geq \frac{1}{p} \left( \mathcal{E}(f^p, f^p) - (p\lambda)^2 \int_M f^{2p} d\Gamma(\phi, \phi) \right)
\]
By Lemma 2.1 and fixing \(p^2 = (p\lambda)^{-2}/(8C_1)\) in (8), we obtain a cutoff function \(\phi = \phi_{p, \lambda}\) for \(B(x, r) \subset B(x, 2r)\) and \(C > 0\) such that
\[
(p\lambda)^2 \int_M f^{2p} d\Gamma(\phi, \phi) \leq \frac{1}{2} \mathcal{E}(f^p, f^p) + C \frac{(\lambda p)^d_w}{r_{d_w}} \int_M f^{2p} d\mu
\]
By (17) and (18), we obtain (15).

By (9) and the above calculations, there exists \(C' > 0\) such that the cutoff functions \(\phi_{p, \lambda}\) satisfy
\[
\|\phi_{p, \lambda} - \phi_{2p, \lambda}\|_\infty \leq \frac{C'}{p\lambda}
\]
for all \(p \geq 1\), for all \(\lambda \geq 1\), for all \(x \in M\) and for all \(r > 0\). This immediately implies (16).

\[\square\]

**Remark.** Estimates similar to (15), were introduced by Davies in [10, equation (3)] to obtain off-diagonal estimates for higher order (order greater than 2) elliptic operators. Roughly speaking, the generator \(\mathcal{L}\) for anomalous diffusion with walk dimension \(d_w\) behaves like an ‘elliptic operator of order \(d_w\)’. However the theory presented in [10] is complete only when the ‘order’ \(d_w\) is bigger than the volume growth exponent \(d_f\), i.e., in the strongly recurrent case. This is because the
method in [10] relies on a Gagliardo-Nirenberg inequality which is true only in the strongly recurrent setting. We believe that one can adapt the methods of [10] to obtain an easier proof for the strongly recurrent case. However, we will not impose any such restrictions and our proof will closely follow the one in [8].

2.3. Proof of Theorem 1.3: Let $\lambda \geq 1$ and $x \in M$ and $r > 0$. Let $p_k = 2^k$ and let $\psi_k = \lambda \phi_{p_k, \lambda}$, where $\phi_{p_k, \lambda}$ is a cutoff function on $B(x, r) \subset B(x, 2r)$ given by Proposition 2.3. We write

$$f_{t,k} := P_t^{\psi_k} f$$

for all $k \in \mathbb{N}$, where $f \in \mathcal{F}$ and $P_t^{\psi_k}$ denotes the perturbed semigroup as in (1).

Using (15), there exists $C_0 > 0$ such that

$$\frac{d}{dt} \| f_{t,0} \|_2^2 = -2\mathcal{E} \left( e^{\psi_1} f_{t,0}, e^{-\psi_1} f_{t,0} \right) \leq 2C_0 \lambda^d \| f_{t,0} \|_2^2$$

and

$$\frac{d}{dt} \| f_{t,k} \|_{2p_k}^2 = -2p_k \mathcal{E} \left( e^{\psi_k} f_{t,k}^{2p_k-1}, e^{-\psi_k} f_{t,k} \right) \leq -\mathcal{E} \left( f_{t,k}^{p_k}, f_{t,k}^{p_k} \right) + 2C_0 \left( \frac{\lambda p_k}{r} \right) \| f_{t,k} \|_{2p_k}$$

for all $k \in \mathbb{N}$. By (20), we obtain

$$\| f_{t,0} \|_{p_1} = \| f_{t,0} \|_2 \leq \exp \left( C_0 \lambda^d t/r^d \right) \| f \|_2.$$  

Using (21) and Nash inequality $\mathcal{N}(d_f, d_w)$, we obtain

$$\frac{d}{dt} \| f_{t,k} \|_{2p_k} \leq -\frac{1}{2C_N p_k} \| f_{t,k} \|_{2p_k}^{1+2d_w p_k/d_f} \| f_{t,k} \|_{p_k}^{-2d_w p_k/d_f} + C_0 p_k^{d_w-1} \left( \frac{\lambda}{r} \right) d_w \| f_{t,k} \|_{2p_k}$$

for all $k \in \mathbb{N}$. By (16) and the fact that $P_t$ is a contraction on $L^\infty$, we have

$$\exp(-2C_1/p_k) f_{t,k+1} \leq f_{t,k} \leq \exp(2C_1/p_k) f_{t,k+1}$$

for all $k \in \mathbb{N}_{\geq 0}$. Combining (23) and (24), we obtain

$$\frac{d}{dt} \| f_{t,k} \|_{2p_k} \leq -\frac{1}{C_A p_k} \| f_{t,k} \|_{2p_k}^{1+2d_w p_k/d_f} \| f_{t,k-1} \|_{p_k}^{-2d_w p_k/d_f} + C_0 p_k^{d_w-1} \left( \frac{\lambda}{r} \right) d_w \| f_{t,k} \|_{2p_k}$$

for all $k \in \mathbb{N}^*$, where $C_A = 2C_N \exp(8d_w C_1/d_f)$.

To obtain off-diagonal estimates using the differential inequalities (25) we use the following lemma. The following lemma is analogous to [8, Lemma 3.21] but the statement and its proof is slightly modified to suit our anomalous diffusion context with walk dimension $d_w$. 


Lemma 2.4. Let \( w : [0, \infty) \rightarrow (0, \infty) \) be a non-decreasing function and suppose that \( u \in C^1([0, \infty); (0, \infty)) \) satisfies
\[
-u'(t) \leq -\frac{\epsilon}{p} \left( \frac{t^{(p-2)/\theta \epsilon_p}}{w(t)} \right)^{\theta \epsilon_p} u^{1+\theta \epsilon_p}(t) + \delta p^{d_w-1} u(t)
\] for some positive \( \epsilon, \theta \) and \( \delta, d_w \in [2, \infty) \) and \( p = 2^k \) for some \( k \in \mathbb{N}^* \). Then \( u \) satisfies
\[
u(t) \leq \left( \frac{2p^{d_w}}{\epsilon \theta} \right)^{1/\theta} t^{(1-\theta)/\theta_p w(t)} e^{\delta t/\theta_p}
\]

Proof. Set \( v(t) = e^{-\delta p^{d_w-1} t} u(t) \). By (26), we have
\[
v'(t) = e^{-\delta p^{d_w-1} t} \left( u(t) - \delta p^{d_w-1} u(t) \right) \leq -\frac{\epsilon t^{p-2}}{pw(t)^{\theta \epsilon_p}} e^{\theta \epsilon_p w(t)} v(t)^{1+\theta \epsilon_p}.
\]
Hence
\[
dt (v(t))^{-\theta \epsilon_p} \geq \epsilon t^{p-2} w(t)^{-\theta \epsilon_p} e^{\delta \epsilon_p w(t)}
\]
and so, since \( w \) is non-decreasing
\[
e^{\delta \epsilon_p w(t)} u(t)^{-\theta \epsilon_p} \geq \epsilon t^{p-2} e^{\delta \epsilon_p w(t)} \int_0^t s^{(p-2)} e^{\delta \epsilon_p w(s)} ds.
\] (28)

Note that
\[
\int_0^t s^{(p-2)} e^{\delta \epsilon_p w(s)} ds \geq \frac{t^{p-1}}{p-1} \int_0^t \exp \left( \delta \epsilon_p w(t) - \delta \epsilon_p t \right) \left[ 1 - (1 - p^{-d_w})^{p-1} \right] \geq \frac{t^{p-1}}{2p^{d_w}} \exp \left( \delta \epsilon_p w(t) - \delta \epsilon_p t \right)
\] (29)

In the last line above, we used the bound \( (1 - p^{-d_w})^{p-1} \geq 1 - p^{-d_w}(p-1) \) for all \( p, d_w \geq 2 \). Combining (28) and (29) yields (27). \( \square \)

We now pick \( f \in L^2(M, \mu) \) and \( f \geq 0 \) with \( \|f\|_2 = 1 \). Let \( u_k(t) = \|f_{t,k-1}\|_{p_k} \) and let
\[
w_k(t) = \sup \{ s^{d_f(p_k-2)/(2d_w p_k)} u_k(s) : s \in (0, t] \}.
\]
By (22), \( w_1(t) \leq \exp(2C_0 \lambda^{d_w t/r^{d_w}}) \). Further by (25), \( u_{k+1} \) satisfies (26) with \( \epsilon = 1/C_A, \theta = 2d_w/d_f, \delta = C_0(\lambda/r)^{d_w}, w = w_k \) and \( p = p_k \). Hence by (27),
\[
u_{k+1}(t) \leq (2^{d_w k + 1}/\epsilon \theta)^{1/(\theta p_k)} t^{(1-p_k)/(\theta p_k)} e^{\delta t/p_k} w_k(t).
\]
Therefore
\[
w_{k+1}(t)/w_k(t) \leq (2^{d_w k + 1}/\epsilon \theta)^{1/(\theta p_k)} e^{\delta t/2^k}
\] for \( k \in \mathbb{N}^* \). Hence, we obtain
\[
\lim_{k \to \infty} w_k(t) \leq C_2 e^{\delta t} w_1(t) \leq C_2 \exp(2C_0 \lambda^{d_w t/r^{d_w}})
\]
where $C_2 = C_2(d_w, \epsilon, \theta)$. Since $P_t$ is a contraction on all $L^p(M, \mu)$ for $1 \leq p \leq \infty$, we obtain
\[
\lim_{k \to \infty} u_k(t) = \|P_t \psi_k\|_\infty \leq \frac{C_2}{\lambda^{d_f/2d_w}} \exp(3C_0 \lambda^{d_w} t / r^{d_w}).
\]
where $\psi_k = \lim_{k \to \infty} \psi_k$. Since the above bound holds for all $f \in L^2(M, \mu)$ with $\|f\|_2 = 1$, we have
\[
\|P_t \psi_k\|_{1 \to \infty} \leq \frac{C_2}{\lambda^{d_f/2d_w}} \exp(2C_0 \lambda^{d_w} t / r^{d_w}).
\]
Combining the above, we have
\[
\|P_t \psi_k\|_{1 \to \infty} \leq \frac{C_2}{\lambda^{d_f/2d_w}} \exp(2C_0 \lambda^{d_w} t / r^{d_w}).
\]
Therefore
\[
p_t(x, y) \leq \frac{C_2}{\lambda^{d_f/2d_w}} \exp\left(\frac{C_3}{(C_4 t)^{d_w}} \exp(2C_0 \lambda^{d_w} t / r^{d_w})\right).
\]
for all $x, y \in M$ and for all $r, t > 0$ and $\lambda \geq 1$. If we choose $r = d(x, y)/2$, we have $\psi_k(y) - \psi_k(x) = -\lambda$. This yields
\[
p_t(x, y) \leq \frac{C_3}{(C_4 t)^{d_w}} \exp(C_4 \lambda^{d_w} t / d(x, y)^{d_w} - \lambda).
\]
where $C_3, C_4 > 1$. Assume $\lambda = C_4^{-1/(d_w-1)}(d(x, y)^{d_w} / t)^{1/(d_w-1)} \geq 1$ in the above equation, we obtain
\[
p_t(x, y) \leq \frac{C_3}{(C_4 t)^{d_w}} \exp\left(-\left(\frac{d(x, y)^{d_w}}{C_5 t}\right)^{1/(d_w-1)}\right).
\]
for all $x, y \in M$ and for all $t > 0$ such that $d(x, y)^{d_w} \geq C_4 t$.

If $d(x, y)^{d_w} < C_4 t$, the on-diagonal estimate (5) suffices to obtain the desired sub-Gaussian upper bound.

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References


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