1. Let $\text{cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X) \cdot \mathbb{E}(Y)$ denote the covariance of two random variables $X,Y$.
   (a) Show that $\text{cov}(X,Y) = \mathbb{E}[(X - \mu_X)(Y - \mu_Y)]$.
   (b) Show that $\sigma^2(X + Y) = \sigma^2(X) + \sigma^2(Y) + 2 \text{cov}(X,Y)$.
   (c) Let $X \sim \text{Unif}[-1,1]$. Show that the random variables $X$ and $X^2$ are uncorrelated, but not independent.
   (d) Let $p(X,Y)$ be the correlation coefficient of $X,Y$. Show that $|p(X,Y)| \leq 1$.
      Hint: Imitate the proof of the Cauchy-Schwarz inequality from linear algebra to show that $|\mathbb{E}(X_1X_2)|^2 \leq \mathbb{E}(X_1^2)\cdot\mathbb{E}(X_2^2)$ for any two RV’s $X_1, X_2$. The claim follows from this by a short calculation.

2. Suppose that $X,Y$ are discrete random variables with joint p.m.f. as shown below. Calculate $\text{cov}(X,Y)$ and $p(X,Y)$.
   \[
   \begin{array}{c|ccc}
   X \downarrow Y & 0 & 1 & 2 & 3 \\
   \hline
   1 & 1/15 & 1/15 & 2/15 & 1/15 \\
   2 & 1/10 & 1/10 & 1/5 & 1/10 \\
   3 & 1/30 & 1/30 & 0 & 1/10 \\
   \end{array}
   \]

3. Let $X$ be the number of rolls of a fair die until I see the first six. Next, I choose a sample, with replacement, of size $X$ from an urn with 5 red and 4 green balls. Let $Y$ be the number of green balls in my sample.
   (a) Compute the conditional probability distribution $p_Y|X(y|x)$.
   (b) Compute $\mathbb{E}[Y|X]$.
   (c) Use your answer from part (b) to compute $\mathbb{E}(Y)$.

4. Assume that the random variables $X,Y$ have joint density function
   \[
   f(x,y) = \begin{cases} 
   \frac{2x+y}{4} & \text{if } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 2, \\
   0 & \text{otherwise.}
   \end{cases}
   \]
   a) What is the conditional probability density function of $X$ given that $Y = 1$?
   b) Find the conditional expectation $\mathbb{E}(X|Y = 1)$.

5. Suppose a laser pointer is located at the origin of your coordinate system, and is pointing toward the vertical line $L = \{(x,y) \text{ s.t. } x = 1\}$. Suppose that the angle $X$ between the laser beam and the $x$ axis is a uniform random variable. Calculate the p.d.f. of the $Y$ coordinate of the point on $L$ which the beam points at. \textit{Hint:} Draw a picture and argue that this means that $X \sim \text{Unif}[-\pi/2,\pi/2]$ and $Y = \tan X$.

6. Let $X_1, X_2 \sim \text{Exp}(\lambda)$ be independent. Calculate the p.d.f. of $X_1 + X_2$.

7. Let $X \sim \text{Poisson}(\lambda)$.
   (a) Calculate the moment generating function of $X$.
   (b) Let $Y \sim \text{Poisson}(\mu)$ be independent of $X$. Show that $X + Y \sim \text{Poisson}(\lambda + \mu)$. 

Hint: What is the m.g.f. of $X + Y$? Remember that the m.g.f. uniquely determines the p.m.f.

8*. Let $X$ be a continuous random variable with p.d.f. $f(x)$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ be a strictly increasing function. Compute the p.d.f. of $g(X)$. 