1. Suppose $X, Y$ are two discrete RV’s with joint p.m.f. according to the table below.
(a) Calculate the marginal p.m.f. of $X$ and of $Y$.
(b) Calculate $\mathbb{P}(X^2 + Y < 3)$.
(c) are $X$ and $Y$ independent?

**Table 1.** The joint p.m.f. of $X, Y$

<table>
<thead>
<tr>
<th>$X \downarrow Y \rightarrow$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2</td>
<td>1/12</td>
<td>1/8</td>
<td>1/8</td>
<td>1/12</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1/12</td>
<td>1/9</td>
<td>1/9</td>
</tr>
<tr>
<td>6</td>
<td>1/12</td>
<td>1/12</td>
<td>0</td>
<td>1/9</td>
</tr>
</tbody>
</table>

**Solution:**
(a) We have $p_X(1/2) = 5/12$, $p_X(1) = 11/36$, $p_X(6) = 5/18$,
and $p_Y(0) = 1/6$, $p_Y(1) = 7/24$, $p_Y(2) = 17/72$, $p_Y(3) = 11/36$.
(b) We have
$$\mathbb{P}(X^2 + Y < 3) = \mathbb{P}((X, Y) \in \{(1/2, 0), (1/2, 1), (1/2, 2), (1, 0), (1, 1)\})$$
$$= \frac{1}{12} + \frac{1}{8} + \frac{1}{8} + \frac{1}{12} = \frac{5}{12}$$
(c) Since the p.m.f.’s are never zero, if the RV’s were independent, the joint p.m.f. would never be zero either. But since there are two zeros in the table, this is not the case. Thus, the RV’s are dependent.

2. You have two dice, one with three sides labeled 0, 1, 2 and one with 4 sides, labeled 0, 1, 2, 3. Let $X_1$ be the outcome of rolling the first die, and $X_2$ the outcome of rolling the second. The rolls are independent.
(a) What is the joint distribution of $(X_1, X_2)$
(b) Let $Y_1 = X_1 \cdot X_2$ and $Y_2 = \max\{X_1, X_2\}$. Make a table for the joint distribution function of $(Y_1, Y_2)$.
(c) Compute the marginal distributions of $Y_1, Y_2$. Are $Y_1, Y_2$ independent?

**Solution:**
(a) By independence we have $p(x, y) = (1/3)(1/4) = 1/12$ for all $x \in \{0, 1, 2\}$ and $y \in \{0, 1, 2, 3\}$.
(b)
Table 2. The p.m.f. of $(Y_1, Y_2)$ with the marginals.

<table>
<thead>
<tr>
<th>$Y_1 \downarrow Y_2 \rightarrow$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>$p_{Y_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1/12</td>
<td>1/6</td>
<td>1/6</td>
<td>1/12</td>
<td>1/12</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1/12</td>
<td>0</td>
<td>0</td>
<td>1/12</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>1/6</td>
<td>0</td>
<td>1/6</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1/12</td>
<td>1/12</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>1/12</td>
<td>0</td>
<td>1/12</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1/12</td>
<td>1/12</td>
</tr>
</tbody>
</table>

$(c)$ For the marginal distributions see the margins of the above table. Since $P(Y_1 = 1, Y_2 = 0) = 0 \neq P(Y_1 = 1)P(Y_2 = 0)$, the variables $Y_1$ and $Y_2$ are not independent.

3. Let $X \sim \text{Exp}(1/2)$, $Y \sim \text{Unif}([2, 4])$, and assume that $X$ and $Y$ are independent. Calculate $P(Y - X \geq 1/2)$.

Solution: The joint density function is

$$f(x, y) = f_X(x)f_Y(y) = \begin{cases} \frac{1}{2}e^{-\frac{1}{2}x} & \text{if } x > 0 \text{ and } 2 < y < 4, \\ 0 & \text{otherwise}. \end{cases}$$

Let $T$ be defined by

$$T = \{(x, y) : x > 0, \ 2 < y < 4, \ x \leq y - 1/2\},$$

then we have

$$P(Y - X \geq 1/2) = \int_T f(x, y) \, dy \, dx = \int_2^4 \int_0^{y - \frac{1}{2}} \frac{1}{4}e^{-\frac{1}{2}x} \, dx \, dy = \frac{1}{2} \int_2^4 \left(1 - e^{-y/2 + 1/4}\right) \, dy = 1 + e^{-1/4} - e^{-7/4} = 1 + e^{-7/4} - e^{-\frac{7}{2}}.$$  

4. Suppose that $X_1, \ldots, X_n$ are independent continuous random variables that all have the same c.d.f. $F(x)$. Define the random variable

$$Y = \max\{X_1, \ldots, X_n\}.$$
Compute the c.d.f. and the p.d.f. of $Y$. Your answer should be in terms of $F(x)$.

**Hint:** Express an inequality of the kind $\max\{X_1, \ldots, X_n\} \leq b$ in terms of separate inequalities for each $X_i$.

**Solution:**
Since $\max\{X_1, \ldots, X_n\} \leq x$ is equivalent to $\{X_1 \leq x\} \cap \cdots \cap \{X_n \leq x\}$, we can use independence and the definition of $Y$ to obtain

$$F_Y(x) = \mathbb{P}(Y \leq x) = \mathbb{P}(X_1 \leq x, \ldots, X_n \leq x) = \prod_{i=1}^{n} \mathbb{P}(X_i \leq x) = (F(x))^n.$$  

We can now get the p.d.f. by differentiation. The chain rule gives

$$f_Y(x) = F'_Y(x) = n F(x)^{n-1} F'(x) = n F(x)^{n-1} f(x).$$

5. The random variables $X, Y$ have joint probability density function

$$f(x, y) = \begin{cases} C y e^{-y-x/y} & \text{if } x > 0 \text{ and } y > 0, \\ 0 & \text{otherwise}. \end{cases}$$

(a) What is the value of $C$? Hint: Integrate with respect to $x$ first.
(b) Find the marginal probability density function $f_Y$.
(c) Compute $\mathbb{P}(X \leq Y^2)$.
(d)* Compute $\mathbb{P}(X \leq Y^3)$

**Solution:**
(a) Using that $\int_{0}^{\infty} y^2 e^{-y} \, dy = 2$ we obtain that

$$1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy = C \int_{0}^{\infty} \int_{0}^{\infty} y e^{-y-x/y} \, dx \, dy = C \int_{0}^{\infty} y^2 e^{-y} \, dy = 2 C,$$

thus $C = 1/2$.

(b) For $y > 0$ we have

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y) \, dx = \frac{1}{2} y e^{-y} \int_{0}^{\infty} e^{-x/y} \, dx = \frac{1}{2} y^2 e^{-y},$$

and $f_Y(y) = 0$ if $y \leq 0$.

(c) We have

$$\mathbb{P}(X \leq Y^2) = \frac{1}{2} \int_{0}^{\infty} \int_{0}^{y^2} y e^{-y-x/y} \, dx \, dy$$

$$= \frac{1}{2} \int_{0}^{\infty} y^2 e^{-y} (1 - e^{-y}) \, dy = \frac{7}{8}.$$
(d) We have
\[ P(X \leq Y^3) = \frac{1}{2} \int_0^\infty \int_0^{y^3} ye^{-y-x/y} \, dx \, dy \]
\[ = \frac{1}{2} \int_0^\infty y^2 e^{-y} - y^2 e^{-y-y^2} \, dy \]
\[ = 1 - \frac{1}{2} \int_0^\infty y^2 e^{-y-y^2} \, dy \]
\[ = 1 - \frac{1}{2} e^{\frac{1}{4}} \int_0^\infty [y(y + \frac{1}{2}) - \frac{1}{2}(y + \frac{1}{2}) + \frac{1}{4}] e^{-(y+\frac{1}{2})^2} \, dy \]
\[ = 1 - \frac{1}{2} e^{\frac{1}{4}} \int_0^\infty [\frac{1}{2} ye^{-(y+\frac{1}{2})^2} + \frac{1}{4} e^{-(y+\frac{1}{2})^2}] \int_0^\infty e^{-(y+\frac{1}{2})^2} \, dy \]
\[ = 1 - \int_0^\infty e^{-(y+\frac{1}{2})^2} \, dy \]
\[ = 1 - \frac{1}{2} \sqrt{\pi} \Phi(X > 0), \]
where \( X \sim \mathcal{N}(-1/2, 1/2) \). Since \( P(X > 0) = 1 - \Phi(1/\sqrt{2}) \), we get
\[ P(X \leq Y^3) = \frac{9}{8} - \frac{1}{8} e^{\frac{1}{4}} (1 - \Phi(1/\sqrt{2})) = 0.92 \]

6. Let \( Z_1 \) and \( Z_2 \) be two points chosen uniformly from the unit disk, independently of each other. Let \( d(Z_1, Z_2) \) denote their Euclidean distance, that is, if \( z_i = (x_i, y_i) \), then \( d(z_1, z_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \). Compute \( E(d(Z_1, Z_2)^2) \).

**Solution:** Since the points are chosen uniformly, the p.d.f. of \( Z_i, i = 1, 2 \) is
\[ f_i(x, y) = \begin{cases} \frac{1}{\pi} & x^2 + y^2 \leq 1 \\ 0 & \text{otherwise} \end{cases} \]
By independence, the joint p.d.f. of \( (Z_1, Z_2) \) is
\[ f((x_1, y_1), (x_2, y_2)) = \begin{cases} \frac{1}{\pi^2} & x_1^2 + y_1^2 \leq 1 \text{ and } x_2^2 + y_2^2 \leq 1 \\ 0 & \text{otherwise} \end{cases} \]
We compute
\[ E(d(Z_1, Z_2)^2) = \frac{1}{\pi^2} \int_{x_1^2 + y_1^2 \leq 1} \int_{x_2^2 + y_2^2 \leq 1} \left[(x_1 - x_2)^2 + (y_1 - y_2)^2\right] dx_1 dy_1 dx_2 dy_2 \]
\[ = \frac{1}{\pi^2} \int_{x_1^2 + y_1^2 \leq 1} \int_{x_2^2 + y_2^2 \leq 1} \left[x_1^2 + x_2^2 + y_1^2 + y_2^2 - 2x_1x_2 - 2y_1y_2\right] dx_1 dy_1 dx_2 dy_2 \]
\[ = \frac{4}{\pi^2} \int_{x_1^2 + y_1^2 \leq 1} \int_{x_2^2 + y_2^2 \leq 1} x_1^2 \, dx_1 dy_1 dx_2 dy_2, \]
where we have used symmetry. We calculate

\[
\begin{align*}
\int\int_{x_1^2+y_1^2 \leq 1} \int\int_{x_2^2+y_2^2 \leq 1} x_1^2 \, dx_1 \, dy_1 \, dx_2 \, dy_2 &= \pi \int\int_{x_1^2+y_1^2 \leq 1} x_1^2 \, dx_1 \, dy_1 \\
&= \pi \int_{-1}^{1} x_1^2 \int_{-\sqrt{1-x_1^2}}^{\sqrt{1-x_1^2}} dy_1 \, dx_1 \\
&= \pi \int_{-1}^{1} 2x_1^2 \sqrt{1-x_1^2} = \frac{\pi^2}{4}
\end{align*}
\]

and conclude \(E(d(Z_1, Z_2)^2) = 1\).

**Second solution:** Denote \(E(d(Z_1, Z_2)^2) = E(X_1^2 + X_2^2 + Y_1^2 + Y_2^2 - 2X_1X_2 - 2Y_1Y_2) = 4E(X_1^2)\), by symmetry, and

\[
E(X_1^2) = \int x_1^2 f_{X_1}(x_1) \, dx_1 = \frac{1}{\pi} \int_{-1}^{1} 2x_1^2 \sqrt{1-x_1^2} \, dx_1 = \frac{1}{4}
\]