1. It is known that 4% of the circuit boards from a production line are defective. If a random sample of 150 circuit boards is taken from this production line, use the Poisson approximation of the Binomial r.v. to estimate the probability that the sample contains:
   a) Exactly 2 defective boards.
   b) At least 2 defective boards.

   **Solution:** The expectation of the number of defective items and of the Poisson distribution should be the same, so \( \lambda = 150 \cdot 0.04 = 6 \). Let \( X \sim \text{Poi}(6) \).

   a) \( P(X = 2) = \frac{6^2}{2!} e^{-6} \approx 4.5\% \).

   b) \( P(X \geq 2) = 1 - P(X = 0) - P(X = 1) = 1 - e^{-6} - 6e^{-6} \approx 98\% \).

2. Let \( X \) be a Poisson random variable with parameter \( \lambda \)
   a) Which \( n = n(\lambda) \geq 0 \) is the most likely value of \( X \), i.e. maximizes \( P(X = n) \)?
   b) Suppose the experiment described by \( X \) has returned the value \( n \geq 0 \). Which parameter \( \lambda = \lambda(n) \) maximizes \( P(X = n) \)?

   **Solution:** a) Let \( g(n) = P(X = n) = e^{-\lambda} \frac{\lambda^n}{n!} \). Then we have
   \[
   \frac{g(n)}{g(n-1)} = \frac{\lambda}{n}.
   \]
   so \( g(n) > g(n-1) \) if and only if \( n < \lambda \) and \( g(n) = g(n-1) \) if \( n = \lambda \) is integer.
   Therefore the probability \( g(n) \) is maximal if \( n \leq \lambda \leq n + 1 \). If \( \lambda \) is an integer then we have two solutions \( n = \lambda \) and \( n = \lambda - 1 \), otherwise \( n = \lfloor \lambda \rfloor \), where \( \lfloor x \rfloor \) denotes the largest integer not greater than \( x \).

   b) In order to find the monotonicity of the function \( f(\lambda) = P(X = n) = e^{-\lambda} \frac{\lambda^n}{n!} \), we need to check the sign of its derivative. We have
   \[
   f'(\lambda) = \frac{e^{-\lambda} \lambda^{n-1}}{n!} (n - \lambda).
   \]
   This gives that \( f'(x) > 0 \) if \( \lambda < n \) and \( f'(x) < 0 \) if \( \lambda > n \), so \( f \) is increasing on the interval \((0, n]\) and decreasing on \([n, \infty)\). Thus \( f \) has a (global) maximum at \( \lambda = n \).

3. Exercise 3.7.

   **Solution:** a) We have \( P(a \leq X \leq b) = F(b) - F(a) = 1 \), which is equivalent to \( F(b) = 1 \) and \( F(a) = 0 \). We need to find the maximal \( a \) and minimal \( b \) with the above property, which gives \( a = \sqrt{2} \) and \( b = \sqrt{3} \). Hence the smallest such interval is \([a, b] = [\sqrt{2}, \sqrt{3}]\).

   b) \[
   P(X = 1.6) = P(X \leq 1.6) - P(X < 1.6) \\
   = P(X \leq 1.6) - \lim_{b \to 1.6-} P(X \leq b) \\
   = F(1.6) - \lim_{b \to 1.6-} F(b) = 0.
   \]
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c) \[ P(1 \leq X \leq 3/2) = P(X \leq 3/2) - P(X < 1) \]
\[ = P(X \leq 3/2) - \lim_{b \to 1^-} P(X \leq b) \]
\[ = F(3/2) - \lim_{b \to 1^-} F(b) \]
\[ = F(3/2) - F(1) \]
\[ = \left(\frac{9}{4} - 2\right) - 0 = \frac{1}{4}. \]

d) \[ f(x) = F'(x) = \begin{cases} 
2x & \text{if } \sqrt{2} < x \leq \sqrt{3}, \\
0 & \text{otherwise}.
\end{cases} \]

Note that actually \( F(x) \) is not differentiable at \( x = \sqrt{2} \) and \( \sqrt{3} \), we can define \( f(x) \) arbitrarily there, it won’t change the integrals.

4. a) Define the function
\[ f(x) = \begin{cases} 
3x - b & x \in [0, 1] \\
0 & \text{otherwise}
\end{cases} \]

Show that there is no value of \( b \) for which this is the p.d.f. of some r.v. \( X \).

b) Let
\[ f(x) = \begin{cases} 
\sin x & x \in [0, b] \\
0 & \text{otherwise}
\end{cases} \]

Show that there is exactly one value of \( b \) for which this could be the p.d.f. of some r.v. \( X \).

Solution: a) First \( f(x) \geq 0 \) for all \( x \in \mathbb{R} \), so \( b \leq 0 \). We also need \( \int_{-\infty}^{\infty} f(x) \, dx = 1 \), so
\[ 1 = \int_{0}^{1} (3x - b) \, dx = \frac{3}{2} - b. \]

Thus we have \( b = \frac{1}{2} \) which does not satisfy \( b \leq 0 \), so \( f \) is not a density function for any \( b \).

b) Again, we require \( f(x) \geq 0 \) for all \( x \in \mathbb{R} \), so \( 0 \leq b \leq \pi \). We also need \( \int_{-\infty}^{\infty} f(x) \, dx = 1 \), so
\[ 1 = \int_{0}^{b} \sin x \, dx = 1 - \cos b. \]

Thus \( \cos b = 0 \) and \( b \in [0, \pi] \), so \( b = \pi/2 \).

5. Answer questions (a) - (f) of Exercise 3.31 for the function
\[ f(x) = \begin{cases} 
 cx^{-3} & x \geq 1 \\
0 & \text{otherwise}
\end{cases} \]

Solution: a) \( f(x) \geq 0 \) is satisfied if \( c \geq 0 \). We need
\[ 1 = \int_{-\infty}^{\infty} f(x) \, dx = \int_{1}^{\infty} cx^{-3} \, dx = \frac{-cx^{-2}}{2} \bigg|_{1}^{\infty} = \frac{c}{2}. \]
so \( c = 2 \).

b) \( \mathbb{P}(0.5 < X < 1) = \int_{0.5}^{1} f(x) \, dx = 0 \).

c) \( \mathbb{P}(0.5 < X < 2) = \int_{0.5}^{2} f(x) \, dx = \int_{1}^{2} 2x^{-3} \, dx = -x^{-2}\bigg|_{1}^{2} = \frac{3}{4} \).

d) \( \mathbb{P}(2 < X < 4) = \int_{2}^{4} f(x) \, dx = \int_{2}^{4} 2x^{-3} \, dx = -x^{-2}\bigg|_{2}^{4} = \frac{3}{16} \).

e) \[
F_X(x) = \int_{0}^{x} f(t) \, dt = \begin{cases} 
1 - x^{-2} & \text{if } x \geq 1, \\
0 & \text{if } x < 1.
\end{cases}
\]

f) \[
\mathbb{E}(X) = \int_{-\infty}^{\infty} x f(x) \, dx = \int_{1}^{\infty} 2x^{-2} \, dx = -2x^{-1}\bigg|_{1}^{\infty} = 2.
\]

Similarly
\[
\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 f(x) \, dx = \int_{1}^{\infty} 2x^{-1} \, dx = 3 \log x\bigg|_{1}^{\infty} = \infty.
\]

Thus
\[
\text{Var}(X) = \infty.
\]

This may seem paradoxical, but it is not: The r.v. with p.d.f. \( 2x^{-3} \) is so spread out that it does not have finite variance. Some r.v.’s don’t even have a finite mean! We will learn later in the course how this is to be interpreted.

6. Suppose that a r.v. \( X \) has cumulative distribution function
\[
F(x) = \begin{cases} 
\frac{2}{\pi} \arctan x & x > 0 \\
0 & x \leq 0
\end{cases}
\]

Compute \( \mathbb{E}\left( \frac{1}{\sqrt{1 + X^2}} \right) \).

Solution: We compute the density function
\[
f(x) = F'(x) = \begin{cases} 
\frac{2}{\pi} \frac{1}{1+x^2} & \text{if } x \geq 0, \\
0 & \text{if } x < 0.
\end{cases}
\]

and therefore
\[
\mathbb{E}\left( \frac{1}{\sqrt{1 + X^2}} \right) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{1 + x^2}} f(x) \, dx = \frac{2}{\pi} \int_{0}^{\infty} \frac{1}{(1+x^2)^{\frac{3}{2}}} \, dx = \frac{2}{\pi} \frac{x}{\sqrt{1+x^2}}\bigg|_{0}^{\infty} = \frac{2}{\pi}.
\]


Solution: We have
\[
\mathbb{P}(Y \leq b) = \mathbb{P}(X \geq c-b) = \begin{cases} 
0 & b \leq 0 \\
\int_{c-b}^{c} \frac{1}{c} = \frac{b}{c} & 0 \leq b \leq c \\
1 & c \leq b
\end{cases},
\]

which is the same as \( \mathbb{P}(X \leq b) \). Since the p.d.f. is the derivative of the cdf, also the p.d.f.’s of \( X \) and \( Y \) coincide.

Solution: a) We have $E(X) = 1/\lambda = 1000$, so $X \sim \text{Exp}(1/1000)$. Thus


b) By the no memory property

$$P(X > 2000 | X > 500) = P(X > 1500) = e^{-1500(1/1000)} = e^{-3/2}.$$