1. Copying 1 billion (10^9) bits from a USB drive to your hard disk takes about 0.2 seconds under ideal conditions. However, every bit has a probability of roughly p = 10^{-8} to be copied incorrectly. Using error correcting codes, your PC is able to recognize and correct flawed bits during the copying process. Assume that it takes 0.02 seconds to correct a flawed bit. Use the Poisson approximation to calculate the probability that copying a movie of 2.5 gigabytes takes less than 9 seconds.

**Solution:** Since 2.5 gigabytes are 2 \cdot 10^{10} bits, we can model the number X of flawed bits when copying the movie by a Poisson RV with parameter \( \lambda = 2 \cdot 10^{10} \cdot p = 200 \). The time consumed by the raw copying plus the error correction is then \( T = 4 + 0.02 \cdot X \) seconds. Therefore,
\[
P(T \leq 25) = P(X \leq 250) = \sum_{k=0}^{250} \frac{200^k}{k!} e^{-200}.
\]
We can use the computer to evaluate this number and get approximately 99.97%.

2. Let X be a Poisson random variable with parameter \( \lambda \)

a) Which \( n = n(\lambda) \geq 0 \) is the most likely value of X, i.e. maximizes \( P(X = n) \)?

b) Suppose the experiment described by X has returned the value \( n \geq 0 \). Which parameter \( \lambda = \lambda(n) \) maximizes \( P(X = n) \)?

**Solution:** a) Let \( g(n) = P(X = n) = e^{-\lambda} \frac{\lambda^n}{n!} \). Then we have
\[
\frac{g(n)}{g(n-1)} = \frac{\lambda}{n},
\]
so \( g(n) > g(n-1) \) if and only if \( n < \lambda \) and \( g(n) = g(n-1) \) if \( n = \lambda \) is integer. Therefore the probability \( g(n) \) is maximal if \( n \leq \lambda \leq n+1 \). (If \( \lambda \) is an integer then we have two solutions \( n = \lambda \) and \( n = \lambda - 1 \), otherwise \( n = \lfloor \lambda \rfloor \), where \( \lfloor x \rfloor \) denotes the largest integer not greater than \( x \).)

b) If the function \( f(\lambda) = P(X = n) = e^{-\lambda} \frac{\lambda^n}{n!} \) is maximal at a point \( \lambda > 0 \), then its derivative must be zero at that point, that is,
\[
f'(\lambda) = e^{-\lambda} \frac{n \lambda^{n-1} - \lambda^n}{n!} = 0.
\]
This gives that \( \lambda = n \). Clearly \( f'(x) > 0 \) if \( \lambda < n \) and \( f'(x) < 0 \) if \( \lambda > n \), so \( f \) is increasing on the interval \([0, n]\) and decreasing on \([n, \infty)\). Thus there is really a global maximum at \( \lambda = n \).
3. Suppose that the continuous RV $X$ has c.d.f. given by

$$F(x) = \begin{cases} 
0 & \text{if } x < \frac{1}{\sqrt{2}} \\
5 - 12\sqrt{2}x + 18x^2 - 4\sqrt{2}x^3 & \text{if } \frac{1}{\sqrt{2}} \leq x < \sqrt{2} \\
1 & \text{if } \sqrt{2} \leq x 
\end{cases}$$

(a) Find the smallest interval $[a,b]$ such that $P(a \leq X \leq b) = 1$.
(b) Find $P(0 < X < \frac{1}{2})$.
(c) Find $P(X = 1)$.
(d) Find $P(1 \leq X \leq \frac{3}{2})$.
(e) Find the p.d.f. of $X$.

**Solution:**

(a) Since $P([a,b]) = F(b) - F(a)$ the smallest such interval is $[\frac{1}{\sqrt{2}}, \sqrt{2}]$.

b) Since $\frac{1}{2} \leq \frac{1}{\sqrt{2}}$, $P(0 < X < \frac{1}{2}) = 0$.

c) Since $X$ is a continuous RV, $P(X = 1) = 0$.

Remark: The fact that $X$ is indeed continuous can be read off from the fact that $F$ is continuous.

d) $P(1 \leq X \leq \frac{3}{2}) = F(\frac{3}{2}) - F(1) = \frac{45}{2} - \frac{27}{\sqrt{2}} - 2\sqrt{2}$

e) $f(x) = F'(x) = \begin{cases} 
-12\sqrt{2} + 36x - 12\sqrt{2}x^2 & \text{if } \frac{1}{\sqrt{2}} \leq x < \sqrt{2}, \\
0 & \text{otherwise.}
\end{cases}$

Note that actually $F(x)$ is actually continuously differentiable at $x = \frac{1}{\sqrt{2}}$ and $\sqrt{2}$, but in principle we can define $f(x)$ arbitrarily there, it won’t change the integrals.

4. Define the function

$$f(x) = \begin{cases} 
9x^2 - 4x^3 + b & x \in [0, 1] \\
0 & \text{otherwise}
\end{cases}$$

Show that there is no value of $b$ for which this is the p.d.f. of some RV $X$.

**Solution:** First $f(x) \geq 0$ for all $x \in \mathbb{R}$, so $b \geq 0$. We also need $\int_{-\infty}^{\infty} f(x) \, dx = 1$, so

$$1 = \int_{0}^{1} (9x^2 - 4x^3 + b) \, dx = 2 + b.$$ 

Thus we have $b = -1$ which does not satisfy $b \geq 0$, so $f$ is not a density function for any $b$. 

5. Suppose a continuous RV $X$ has the p.d.f.

$$f(x) = \begin{cases} \frac{c}{1+x^2} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

(a) Find the c.d.f. of $X$.
(b) What must be the value of $c$?
(c) Find $E(X)$.
(d) Compute $E\left(\frac{1}{\sqrt{1+X^2}}\right)$.

**Solution:**
(a) The c.d.f. is the integral of the p.d.f., and so, according to integration tables, $F(x) = c \arctan x + C$ if $x > 0$ and 0 otherwise. The constant of integration has to be $C = 0$, for $F$ to be continuous at 0.

(b) We need $F(\infty) = 1$, and this means that $c^{-1} = \arctan \infty = \frac{\pi}{2}$, or $c = \frac{2}{\pi}$.

(c) By definition of the expectation, we have

$$E(X) = \int_{-\infty}^{\infty} x \cdot \frac{2}{\pi} \frac{1}{1+x^2} \, dx$$

This integral is divergent, and therefore $X$ does not have a well defined average (it is too spread out). We will learn later in the course how this is to be interpreted.

d)

$$E\left(\frac{1}{\sqrt{1+X^2}}\right) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{1+x^2}} f(x) \, dx = \frac{2}{\pi} \int_{0}^{\infty} \frac{1}{(1+x^2)^{3/2}} \, dx = \frac{2}{\pi} \left[ \frac{x}{\sqrt{1+x^2}} \right]_{0}^{\infty} = \frac{2}{\pi}.$$

6. Let $c > 0$ and $X \sim \text{Unif}[0,c]$. Show that the RV $Y = c - X$ has the same c.d.f. and therefore also the same p.d.f. as $X$.

**Solution:** We have

$$P(Y \leq b) = P(X \geq c - b) = \begin{cases} 0 & b \leq 0 \\ \int_{c-b}^{c} \frac{1}{c} \, dx = \frac{b}{c} & 0 \leq b \leq c \\ 1 & c \leq b \end{cases},$$

which is the same as $P(X \leq b)$. Since the p.d.f. is the derivative of the cdf, also the p.d.f.'s of $X$ and $Y$ coincide.

7. (a) Suppose that the duration $T$ (in hours) of your morning routine (breakfast, shower, etc.) is modeled by an exponential RV with parameter $\lambda$. You set your alarm 1 hour before your bus leaves for UBC. For which value of $\lambda$ do you have a 50% chance of catching the bus?

(b)* Calculate the $n$th moment of $T$.

**Solution:**
(a) We are interested for which $\lambda$ we have $P(T \leq 1) = \frac{1}{2}$. We compute

$$P(T \leq 1) = \int_{0}^{1} \lambda e^{-\lambda x} \, dx = 1 - e^{-\lambda},$$

where $\lambda$ is the rate parameter of the exponential distribution.
and so $\lambda = \ln 2$.

b) We have

$$E X^n = \int_0^\infty x^n \cdot \lambda e^{-\lambda x} \, dx = \lambda^{-n} \int_0^\infty x^n e^{-x} \, dx$$

Let us call the latter integral $I_n$. Using integration by parts, we have

$$I_n = \left[ -x^n e^{-x} \right]_0^\infty + nI_{n-1} = nI_{n-1}$$

if $n \geq 1$. Since $I_0 = 1$ by the normalization property (or computation), we conclude that $I_n = n!$, and $E X^n = \lambda^{-n} n!$. 