1. Consider the following game: An urn contains 20 white balls and 10 black balls. If you draw a white ball, you get $1, but if you draw a black ball, you lose $2.
(a) You draw 6 balls out of the urn. What is the probability that you will win money?
(b) How many balls should you draw in order to maximize the probability of winning? \textit{Hint:} Use a computer.

2. In a group of 5 teenagers, what is the probability that at least two of them were born in the same year?

3. Assume that the events $E_1, E_2$ are independent.
   a) Prove that the events $E_1^c, E_2^c$ are also independent.
   b) If, in addition, $P(E_1) = \frac{1}{2}$ and $P(E_2) = \frac{1}{4}$, prove that $P(E_1 \cup E_2) = \frac{2}{3}$.
   c) If, in addition, $E_3$ is a third event that is independent of $E_1$ and of $E_2$, and such that $P(E_3) = \frac{1}{4}$, prove that $\frac{17}{24} \leq P(E_1 \cup E_2 \cup E_3) \leq \frac{19}{24}$.

4. Eight rooks are placed randomly on a chess board. What is the probability that none of the rooks can capture any of the other rooks? (In non-chess terms: Randomly pick 8 unit squares from an $8 \times 8$ square grid. What is the probability that no two squares share a row or a column?)
   \textit{Hint:} How many choices do you have to place rooks in the first row? After you have made your choice, how many choices do you have for the second? Continue this reasoning.

5. We toss two dice. Consider the events
   E: The sum of the outcomes is odd.
   F: At least one outcome is 4.
   Calculate the conditional probabilities $P(E \mid F)$ and $P(F \mid E)$.

6. A fair die is rolled repeatedly.
   (a) Give an expression for the probability that the first five rolls give a three at most two times.
   (b) Calculate the probability that the first three does not appear before the fifth roll.
   (c) Calculate the probability that the first three appears before the twentieth roll, but not before the fifth roll.

7.* Let the sequence of events $E_1, E_2, \ldots, E_n$ be independent, and assume that $P(E_i) = \frac{1}{i+1}$. Show that $P(E_1 \cup \cdots \cup E_n) = \frac{n}{n+1}$. 