Suggested Exercises I:

1. Let \( U = (u_1, \ldots, u_n) \in \mathbb{R}^n \) (viewed as a function on \( \{1, \ldots, n\} \)). Let \( \beta \in \mathbb{R} \). Let \( \mu_\beta \) be the probability vector \( (p_1, \ldots, p_n) \) defined by
   \[
   p_i = \frac{e^{-\beta u_i}}{Z(\beta)}
   \]
   where \( Z \) is the normalization factor:
   \[
   Z(\beta) = \sum_j e^{-\beta u_j}.
   \]
   Show that
   \[
   \frac{dE_{\mu_\beta}(U)}{d\beta} = -\text{Variance}_{\mu_\beta}(U).
   \]

2. Show that
   \( I(x) = -\log p(x) \) is the unique function s.t.
   a. \( I(x) \geq 0 \) and \( I(x) \not\equiv 0 \).
   b. \( I(x) = I(p(x)) \).
   c. If \((X,Y)\) are jointly distributed and the events \((X = x)\) and \((Y = y)\) are independent, then \( I(x,y) = I(x) + I(y) \).

3. Recall some of our beloved entropy properties:
   i. \( H(Y|X) \leq H(Y|f(X)) \)
   ii. \( H(X,Y|Z) = H(X|Z) + H(Y|X,Z) \)
   iii. \( H(Y|Z,X) \leq H(Y|Z) \) with equality iff \( X \perp_Z Y \)
   iv. If \( X \perp_Z (Y,W) \), then \( H(Y|W,Z,X) = H(Y|W,Z) \).
   v. \( H(f(X)|Y) \leq H(X|Y) \)
   (these were called properties 8 - 12).
   Prove properties ii - v.

4. 
   - When is there equality in property i?
   - Is the converse to property iv true?
   - When is there equality in property v?

5. For a stationary finite-state Markov chain, must the sequence \( \frac{H(X_1, \ldots, X_n)}{n} \) stabilize?

6. Let \( \overline{X} \) be a stationary process. Show that if \( h(\overline{X}) = H(X_2|X_1) \), then \( \overline{X} \) is first-order Markov.

7. Let \( \overline{X} \) be stationary ergodic. For \( \delta < 1/2 \), let \( B^n_\delta \) be any set of sequences of length \( n \) such that for all \( n \), \( \mu(B^n_\delta) > 1 - \delta \). Show that
   \[
   \liminf_{n \to \infty} \frac{\log |B^n_\delta|}{n} \geq h.
   \]