Suggested Exercises I:

1. Let $U = (u_1, \ldots, u_n) \in R^n$ (viewed as a function on $\{1, \ldots, n\}$. Let $\beta \in R$. Let μ_β be the probability vector $(p_1, \ldots, p_n)$ defined by

$$p_i = \frac{e^{-\beta u_i}}{Z(\beta)}$$

where Z is the normalization factor:

$$Z(\beta) = \sum_j e^{-\beta u_j}.$$

Show that

$$\frac{dE_{\mu_\beta}(U)}{d\beta} = -\text{Variance}_{\mu_\beta}(U).$$

2, Show that
$I(x) = -\log p(x)$ is the unique function s.t.

a. $I(x) \geq 0$ and $I(x) \not\equiv 0$.

b. $I(x) = I(p(x))$.

c. If (X, Y) are jointly distributed and the events $(X = x)$ and $(Y = y)$ are independent, then $I(x, y) = I(x) + I(y)$.

3.
Recall some of our beloved entropy properties:

i $H(Y|X) \leq H(Y|f(X))$

ii $H(X, Y|Z) = H(X|Z) + H(Y|X, Z)$

iii $H(Y|Z, X) \leq H(Y|Z)$ with equality iff $X \perp_Z Y$

iv If $X \perp_Z (Y, W)$, then $H(Y|W, Z, X) = H(Y|W, Z)$.

v $H(f(X)|Y) \leq H(X|Y)$

(these were called properties 8 - 12).

Prove properties ii - v.

4.

– When is there equality in property i?

– Is the converse to property iv true?

– When is there equality in property v?

5. For a stationary finite-state Markov chain, must the sequence $\frac{H(X_1,\ldots,X_n)}{n}$ stabilize?

6. Let $\overline{X}$ be a stationary process. Show that if $h(\overline{X}) = H(X_2|X_1)$, then $\overline{X}$ is first-order Markov.

7, Let $\overline{X}$ be stationary ergodic. For $\delta < 1/2$, let B_δ^n be any set of sequences of length n such that for all n, $\mu(B_\delta^n) > 1 - \delta$. Show that

$$\liminf_{n\to\infty} \frac{\log |B_\delta^n|}{n} \geq h.$$