Suggested Exercises III:

1. Deduce the Weil equidistribution theorem from the ergodic theorem (fill in the details of the argument given in class):

   If \( \alpha \in [0, 1] \) is irrational, then \( \{\alpha, 2\alpha, 3\alpha, \ldots, \} \) is uniformly distributed mod 1 in \([0, 1]\):
   for any subinterval \((a, b) \subset [0, 1]\)
   \[
   \frac{|\{i \in [0, n-1] : i\alpha \in (a, b)\}|}{n} \to b - a
   \]

2. Show that mixing is an invariant of isomorphism of MPT’s.

3. Complete the proof that a stochastic matrix is primitive iff it is irreducible and aperiodic.

4. Complete the geometric proof (using convexity ideas, given in class) that if \( P \) is a primitive stochastic matrix with stationary vector \( \pi \), then
   \[
   (P^n)_{ij} \to \pi_j
   \]

5. Show that if \( \alpha \) is a (finite, measurable) partition and \( \mathcal{B}_n, \mathcal{B} \) are sigma-algebras s.t. \( \mathcal{B}_n \uparrow \mathcal{B} \), then
   \[
   H(\alpha|\mathcal{B}_n) \downarrow H(\alpha|\mathcal{B}).
   \]

In the following problems, let \( \alpha \) and \( \beta \) be finite measurable partitions and \( \mathcal{E}, \mathcal{F} \) sub-sigma-algebras, and \( \sigma(\cdot) \) be the sigma-algebra generated by \( \cdot \).

6. Show that \( I_{\alpha \vee \beta}, \mathcal{F} = I_\alpha, \mathcal{F} + I_\beta, \sigma(\alpha, \mathcal{F}) \) a.e.

7. Show that \( H(\alpha \vee \beta|\mathcal{F}) = H(\alpha|\mathcal{F}) + H(\beta|\sigma(\alpha \cup \mathcal{F})) \leq H(\alpha|\mathcal{F}) + H(\beta|\mathcal{F}) \)

8. Show that \( H(\beta|\sigma(\mathcal{F} \cup \mathcal{E})) \leq H(\beta|\mathcal{F}) \) with equality iff \( \mathcal{E} \perp \mathcal{F} \beta \)

9. Does \( \beta \preceq \alpha \) imply:
   (a) \( I_{\beta|\mathcal{F}} \leq I_{\alpha|\mathcal{F}} \) a.e. -or-
   (b) \( I_{\beta|\mathcal{F}} \geq I_{\alpha|\mathcal{F}} \) a.e. -or-
   (c) neither

10. Does \( \mathcal{E} \subseteq \mathcal{F} \) imply
   (a) \( I_{\alpha|\mathcal{F}} \leq I_{\alpha|\mathcal{E}} \) a.e. -or-
   (b) \( I_{\alpha|\mathcal{F}} \geq I_{\alpha|\mathcal{E}} \) a.e. -or-
    (c) neither