Suggested Exercises III:

1. Deduce the Weil equidistribution theorem from the ergodic theorem (fill in the details of the argument given in class) :

   If $\alpha \in [0,1]$ is irrational, then $\{\alpha, 2\alpha, 3\alpha, \ldots, \}$ is uniformly distributed mod 1 in $[0,1]$: for any subinterval $(a,b) \subset [0,1]$

$$\frac{|\{i \in [0, n-1] : i\alpha \in (a,b)\}|}{n} \to b - a$$

2. Show that mixing is an invariant of isomorphism of MPT's.

3. Complete the proof that a stochastic matrix is primitive iff it is irreducible and aperiodic.

4. Complete the geometric proof (using convexity ideas, given in class) that if P is a primitive stochastic matrix with stationary vector π, then

$$(P^n)_{ij} \to \pi_j$$

5. Show that if α is a (finite, measurable) partition and $\mathcal{B}_n, \mathcal{B}$ are sigma-algebras s.t. $\mathcal{B}_n \uparrow \mathcal{B}$, then

$$H(\alpha|\mathcal{B}_n) \downarrow H(\alpha|\mathcal{B}).$$

   In the following problems, let α and β be finite measurable partitions and $\mathcal{E}, \mathcal{F}$ sub-sigma-algebras, and $\sigma(\cdot)$ be the sigma-algebra generated by $\cdot$

6. Show that $I_{\alpha\vee\beta|\mathcal{F}} = I_{\alpha|\mathcal{F}} + I_{\beta|\sigma(\alpha\cup\mathcal{F})}$ a.e.

7. Show that $H(\alpha \vee \beta|\mathcal{F}) = H(\alpha|\mathcal{F}) + H(\beta|\sigma(\alpha \cup \mathcal{F})) \leq H(\alpha|\mathcal{F}) + H(\beta|\mathcal{F})$

8. Show that $H(\beta|\sigma(\mathcal{F} \cup \mathcal{E})) \leq H(\beta|\mathcal{F})$ with equality iff $\mathcal{E} \perp_{\mathcal{F}} \beta$

9. Does $\beta \preceq \alpha$ imply:

   (a) $I_{\beta|\mathcal{F}} \leq I_{\alpha|\mathcal{F}}$ a.e. –or–
   (b) $I_{\beta|\mathcal{F}} \geq I_{\alpha|\mathcal{F}}$ a.e. –or–
   (c) neither

10. Does $\mathcal{E} \subseteq \mathcal{F}$ imply

   (a) $I_{\alpha|\mathcal{F}} \leq I_{\alpha|\mathcal{E}}$ a.e. –or–
   (b) $I_{\alpha|\mathcal{F}} \geq I_{\alpha|\mathcal{E}}$ a.e. –or–
   (c) neither