Axiom A: Diffeomorphism

Let \((M,g)\) be a compact \(C^\infty\) Riemannian manifold. Also let \(f: M \to M\) be a diffeomorphism.

\[
\text{df: } TM \to TM, \quad TM = U_x T_x M, \quad \text{df}_x: T_x M \to T_{f(x)} M
\]

\textbf{Hyperbolicity:} A closed subset \(N \subset M\) is called \textit{hyperbolic} if

\[
f(N) = N \quad \text{and} \quad \forall x \in N, \quad T_x M = E^s_x \oplus E^u_x
\]

\begin{enumerate}
  \item \(Df(E^s_x) = E^s_{f(x)}\), \(Df(E^u_x) = E^u_{f(x)}\)
  \item \(\exists c > 0 \text{ and } d \in (0,1)\)
    \[
    \|Df^n(x)\|_g \leq c d^{n+1}\|x\|_g, \quad \forall x \in E^s_x, \ n \geq 0.
    \]
    \[
    \|Df^{-n}(x)\|_g \leq c d^{n+1}\|x\|_g, \quad \forall x \in E^u_x, \ n \geq 0
    \]
  \item \(E^s_x \& E^u_x\) are continuous in \(x\), i.e.
    \[
    E^u_x = U_{x \in N} \text{ and } T_NM = U_{x \in N} T_x M
    \]
    \[
    T_NM = E^u_x \oplus E^s_x 
    \]
    \(\Pi: E \to M\) is cont.
\end{enumerate}

\textbf{Example:} \(X^2 \to W^{+}\) Let \(f^2\) be the 2-dim hyperbolic manifold, \(f_t\) is the geodesic flow which is Anosov flow.

\[
Tf^2 \cong \mathbb{PSL}_2(\mathbb{R})
\]

\[
T_x f^2 = W^u(x) \oplus g_t(x) \oplus W^s(x)
\]

This is a splitting for a hyperbolic geodesic flow.
Let, \( x \in M \), \( \forall u \exists x, s.t.: \)

\[ \cup u \text{ or } f^n(u) \neq \emptyset \Rightarrow x \text{ is non-wandering} \]

\[ \cup u \text{ or } f^n(u) \neq \emptyset \Rightarrow x \text{ is periodic} \]

Let, \( \Omega(f) \) is set of non-wandering points of \( f \).

Define: \( f \) satisfies \textbf{Axiom A} if, \( \Omega(f) \) is hyperbolic and \( \Omega(f) = \{ x | x \text{ is periodic and } x \neq f \} \).

We call \( f \) to be \textit{Anosov} if \( \Omega(f) = M \) and if \( f \) is hyperbolic.

Fact 1: It may not be true that \( \Omega(f) = M \), for an \( f \) which is Anosov.

4) The condition (i) in the definition of hyperbolicity does not depend on \( g \), however, \( c, d \) depend on \( g \). We always assume \( g \) is such a metric s.t. \( c > 1 \) for \( \Omega(f) \) where \( f \) is (Anosov) Axiom satisfies Axiom A.

5) Every \( f \) which satisfies the Axiom A has such a metric.
For $x \in M$,
\[ W^s(x) = \{ y \in M : d(f^n(x), f^n(y)) \to 0 \text{ as } n \to \infty \} \text{ stable manifold} \]
\[ W^c_+(x) = \{ y \in M : d(f^n(x), f^n(y)) \not\to c, \forall n > 0 \} \text{ unstable manifold} \]
\[ W^s_+(x) = \{ y \in M : d(f^n(x), f^n(y)) \to 0 \text{ as } n \to 0 \} \text{ stable manifold} \]
\[ W^c_-(x) = \{ y \in M : d(f^n(x), f^n(y)) < e, \forall n > 0 \} \text{ unstable manifold} \]

**Theorem**: Let $N$ be a hyperbolic set for a $C^1$-diffeomorphism $f$. For $\epsilon > 0$ sufficiently small

a) $W^s_+(x), W^u_+(x)$ are $C^n$-disks for $x \in N$, with
\[ T_x W^s_+(x) = E^s_x \text{ and } T_x W^u_-(x) = E^u_x \]

b) $d(f^n(x), f^n(y)) \leq d(x, y)^n$ for $y \in W^s_+(x)$, $n \geq 0$.
\[ d(f^n(x), f^n(y)) \leq d(x, y)^n \text{ for } y \in W^u_-(x), n \geq 0. \]

c) $W^s_+(x), W^u_-(x)$ vary continuously with $x$ in $C^2$ topology.

**Corollary**: For $x \in N \subseteq M$ hyperbolic
\[ W^s_+(x) \cap W^s_-(x) = \bigcup_{n \geq 0} f^{-n} W^s_+(f^n(x)) \]
\[ W^u_+(x) \cap W^u_-(x) = \bigcup_{n \geq 0} f^n W^u_-(f^{-n}(x)) \]
Canonical Coordinates

def. f satisfies Axiom A. For any small ε > 0

\exists \delta > 0 \ s.t. \ W^s_ε(x) \cap W^u_ε(y) \text{ consists of a single}

point z, which we denote by [x, y], whenever d(x, y) < ε

and, x, y \in \mathcal{Ω}(f).

Furthermore, \ [x, y] \in \mathcal{Ω}(f) \text{ and }

\[ [; ; ] : \{ \mathcal{Ω}(f) \times \mathcal{Ω}(f) \mid d(x, y) \leq \delta \} \to \mathcal{Ω}(f) \text{ is continuous. } \]

Proof sketch:

\[ W^u_ε(x) \cap W^s_ε(y) = \{ x \} \]

because of transversality,

which remains true even after \ f-\ perturbation.

\[ [x, y] \in \mathcal{Ω}(f) \] is a result of the fact that periodic

points are dense in \ \mathcal{Ω}(f).

Cor. Proof:

let N \subseteq M \text{ hyperbolic. Then } \exists \delta > 0 \text{ s.t. } N \text{ is}

expansive in M, i.e. if x \in N \text{ and } y \in M, then

\[ d(f^k x, f^k y) \geq \delta \text{ for some } k \in \mathbb{Z}. \]

Proof:

Otherwise, \ y \in W^s_ε(x) \cap W^u_ε(x) \text{ and,}

by previous theorem, \ x = y, \ contradiction.
Examples.

1) Two-dim forms: Let, \( M = \mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2 \)

\[ \phi : \mathbb{T}^2 \to \mathbb{T}^2 \] \( \Phi \) is a map.

\( (s,t) \to (s+e, s) \mod 1 \) and it has a cont. inverse

\[ \phi^{-1} : \mathbb{T}^2 \to \mathbb{T}^2 \]

\( (s,t) \to (e, s-t) \mod 1 \)

So, \( (\mathbb{T}^2, \Phi) \) is an invertible dynamical system. In particular, \( \Phi \) is an (group) automorphism of \( \mathbb{R}^2 / \mathbb{Z}^2 \)

and is called a 
hyperbolic

Toral automorphism.

Let, \( \pi : \mathbb{R}^2 \to \mathbb{T}^2 \) is the projection map

\( (s,t) \to (s,t) \mod \mathbb{Z}^2 \)

Let, \( A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \) is the linear map on \( \mathbb{R}^2 \).

\[ \begin{array}{ccc}
\mathbb{R}^2 & \xrightarrow{A} & \mathbb{R}^2 \\
\downarrow{\pi} & & \downarrow{\pi} \\
\mathbb{T}^2 & \xrightarrow{\Phi} & \mathbb{T}^2 \\
\end{array} \]

So \( \Phi \) is the defined toral automorphism.

Note that eigenvalues of \( A \) are \( \frac{1 \pm \sqrt{5}}{2} \). One

Perron-Frobenius eigenvalue \( \lambda = \frac{1 + \sqrt{5}}{2} > 1 \) and other one \( \lambda = \frac{1 - \sqrt{5}}{2} < 1 \).

The eigenvalues \( \lambda_1 \) and \( \lambda_2 < 1 \), one dimensional, say \( L_1 \) and \( L_2 \)

So, \( A \) expands in the direction (as they are one-dim direction makes sense) of \( L_1 \) and contracts and flips in the direction of \( L_2 \).
There are similar phenomena in higher dimensional tori. Let $\phi$ be a toral automorphism on $T^n$.

$$\phi : \mathbb{R}^n / \mathbb{Z}^n \to \mathbb{R}^n / \mathbb{Z}^n$$

$x \mapsto Ax + \mathbb{Z}^n$

where $A$ is a linear transform on $\mathbb{R}^n$, s.t. $\phi$ is invertible. A toral automorphism is called hyperbolic if no eigenvalue of $A$ has absolute value one. Similarly, there are no eigenvalues where $A$ expands and no where $A$ contracts.

**Fact Remark:** There exists orbifold (i.e. manifold/stratify) where which are not hyperbolic, i.e. exist neutral direction. Example, $\mathbb{C} \mathbb{H} (\mathbb{R}) / \mathbb{C} \mathbb{H} (\mathbb{Z})$ has $(n-1)$ dimension of flats (neutral directions).
Horseshoe

Let $S$ be a unit square in $\mathbb{R}^2$ and let $\phi$ be a 1-1 matching, $\phi : S \rightarrow \mathbb{R}^2$ s.t. $\phi(S)$ looks like below.

Both restrictions $\phi|_U$ and $\phi|_T$ are linear mappings which contract in the vertical direction and expand in the horizontal direction. $\phi$ is called the horseshoe mapping.

$(S, \phi)$ is not a dynamical system since $\phi(S) \neq S$. We restrict $\phi$ to $M = \bigcap_{n=0}^{\infty} \phi^n(S)$.

Fact: $\phi$ is an Axiom A diffeomorphism, where stable and unstable manifolds intersect, for periodic points.

$M$ is a product of two cantor sets and is maximal invariant set for $\phi$.

Here under Euclidean metric, $2^{\frac{1}{2}}$ and $2^{\frac{1}{2}}$ are hyperbolic splitting for vertical and horizontal direction.
3) **Solenoid**

Let $M = S^1 \times D$ be the solid torus. Let’s define

$$f : M \to M$$

$$(\phi, x, y) \mapsto (2\phi, \frac{1}{10} x + \frac{1}{5} \cos \phi + \frac{1}{5} y + \frac{1}{5} \sin \phi)$$

Then $M_0 = \cap f^n(M)$ is an attractor where $f$ is expanding locally if it is a cantor set x interval. The stable manifolds and the sections $C = \{ \theta \} \times D$ and unstable manifolds are entirely contained in the attractor.

The map $f$ is a smooth embedding of $T$ onto itself the preserves the foliation by (conical disk) cross-sectional disks. The map $f$ expands in the longitudinal direction, contracts cross-sectional disk and then warps the deformed torus tube tube inside $T$ with twisting but without self intersection.

**Properties**

- A $S^1 \times D$ i.e. cross-sectional disks are stable wrt.
- They are cantor set in $M_0$. $f$ is an $A$.
- Periodic points of $f$ are dense.
- $f$ is topologically transitive.
let $f$ be an Axiom A diffeomorphism, $\Omega(f)$ be the set non-wandering points of $f$. Then,

**Theorem**: $\Omega(f) = \bigcup_{i=1}^{s} \Omega_i$, $\Omega_i \cap \Omega_j = \emptyset$, $i \neq j$, closed.

1. $f^{-1}(\Omega_i) = \Omega_i$ and $f|_{\Omega_i}$ is hyperbolic.
2. $\Omega_i = \bigcup_{j \neq i} \Omega_j \times \{j\}$, $\Omega_i \cap \Omega_k = \emptyset$, $i \neq k$, closed

$f(\Omega_i) = \Omega_i \times \{i\}$ and $f^n|_{\Omega_i}$ is topologically mixing.

This theorem is called "Spectral decomposition".

# It was a long-standing question if $f$ is Anosov whether $\Omega(f) = M$ or not. (False).

**Anosov closing lemma**: If $f$ is Anosov diff. ($M$ is hyperbolic), then

if $f$ satisfies Axiom A.

**Proof**: Fundamental Neighbourhood

If $f$ satisfies axiom A, $\exists$ a nbhd $U$ of $\Omega(f)$ s.t.

$$\bigcap_{n \in \mathbb{Z}} f^n(U) = \Omega(f)$$
$$W^s(x) = \{ x \cdot n(a) | a \in \mathbb{R} \}, \quad n(a) = \left( \begin{array}{c} a \\ 1 \end{array} \right)$$

$$W^u(x) = \{ x \cdot n(b) | b \in \mathbb{R} \}, \quad n(b) = \left( \begin{array}{c} b \\ 1 \end{array} \right)$$

$$T \mathbb{R}^2 \supset E^+ \oplus E^0 \oplus E^-$$

A flow $\phi: M \times \mathbb{R} \rightarrow M$.

$\phi(x,0) \rightarrow x$.

$\phi(\phi(x,t), s) = \phi(x, t+s)$.

An Anosov flow $\varrho_t: TM \rightarrow TM$.

1. Transversal:
   $$T_x M = T_x (d_{\varrho_t} x) \oplus T_x W^s(x) \oplus T_x W^u(x)$$

2. $g_t W^s(x) = W^s(g_t x)$ (Stability)
   $g_t W^u(x) = W^u(g_t x)$

3. Contraction:
   $$y \in W^s(x)$$
   $$d(g_t x, g_t y) \leq c e^{-t} d(x,y)$$
   $$y \in W^u(x)$$
   $$d(g_t x, g_t y) \leq c e^{-t} d(x,y)$$

**Anosov closing lemma**

Let for some $x \in X$, $t > 0$, $d(x, g_t x) < c$ (c given) then, $\exists$ periodic $x_0 \in X$ s.t. $d(x, x_0) < \varepsilon$