Axiom A Diffeomorphism

Let (M,g) be a compact C^∞-Riemannian manifold. Also let, $f: M \to M$ be a diffeomorphism.

$df: TM \to TM$, $\quad TM = \bigcup_{x\in M} T_xM$, $df_x : T_xM \to T_{f(x)}M$

Hyperbolicity: A closed subset $N \subseteq M$ is called <u>hyperbolic</u> if $f(N) = N$ and $\forall x \in N$, $T_xM = E^u_x \oplus E^s_x$ s.t.

(i) $Df(E^s_x) = E^s_x$, $\quad Df(E^u_x) = E^u_x$

(ii) $\exists\, c>0$ and $\lambda \in (0,1)$

$$\|Df^n(v)\|_g \leq c\lambda^n \|v\|_g, \quad v \in E^s_x, \; n \geq 0.$$

$$\|Df^{-n}(v)\|_g \leq c\lambda^n \|v\|_g, \quad v \in E^u_x, \; n \geq 0$$

(iii) E^s_x & E^u_x are continuous in x. i.e.

$E^u = \bigcup_{x\in N} E^u_x$ is <u>continuous</u> subbundle of $T_NM = \bigcup_{x\in N} T_xM$ $\longrightarrow$ $\pi : E \to M$ is cont.

and $T_NM = E^u \oplus E^s$.

Example: (*) ~~$T_2\mathbb{H} = W^s$~~ — Let, $\mathbb{H}^2$ be the 2-dim hyperbolic manifold. g_t is the geodesic flow which is Anosov flow.

$T\mathbb{H}^2 \cong PSL_2(\mathbb{R}) =$

$$T_x\mathbb{H}^2 = W^u(x) \oplus g_t(x) \oplus W^s(x)$$

This is a splitting for a hyperbolic geodesic flow.

Let, $x \in M$, $\forall U \ni x$, s.t.

$$U \cap \bigcup_{n>0} f^n(U) \neq \phi \Rightarrow x \text{ is non-wandering}$$

x is periodic

Let, $\Omega(f)$ is set of non-wandering pts of f.

<u>Def^n</u>: f satisfies <u>Axiom A</u> if, $\Omega(f)$ is hyperbolic and $\Omega(f) = \overline{\{x \mid x \text{ is periodic} - f\}}$

We call f to be Anosov if ~~$\Omega(f) = M$ and it is~~ M is hyperbolic.

<u>Fact</u>: i) It may not be true that $\Omega(f) = M$, for an f which is Anosov.

ii) The condition (ii) in the definition of hyperbolicity does not depend on g, (however, c, d depend on g) We ~~always~~ assume g is such a metric s.t. $c > 1$. for $\Omega(f)$ where f ~~is (Anosov) Axiom~~ satisfies Axiom A

iii) Every f which satisfies the Axiom A has such a metric.

For $x \in M$,

$$W^s(x) = \{ y \in M : d(f^n x, f^n y) \to 0 \text{ as } n \to \infty \}$$

$$W^s_\epsilon(x) = \{ y \in M : d(f^n x, f^n y) \leq \epsilon, \ \forall n \geq 0 \}$$

} stable manifold

} ϵ - stable manifold

$$W^u(x) = \{ y \in M : d(f^{-n}(x), f^{-n}(y)) \to 0 \text{ as } n \to \infty \}$$

$$W^u_\epsilon(x) = \{ y \in M : d(f^{-n}(x), f^{-n}(y)) \leq \epsilon, \ \forall n \geq 0 \}$$

} unstable manifold

} ϵ-unstable manifold

<u>Theorem</u>: Let, N be ~~be~~ a hyperbolic set for a C^r-diffeomorphism f. For $\epsilon > 0$ sufficiently small

(a) $W^s_\epsilon(x)$, $W^u_\epsilon(x)$ are C^r-disks for $x \in N$, with $T_x W^s_\epsilon(x) = E^s_x$ and $T_x W^u_\epsilon(x) = E^u_x$.

(b) $d(f^n x, f^n y) \leq \lambda^n d(x, y)$ for $y \in W^s_\epsilon(x)$, $n \geq 0$.
$d(f^{-n} x, f^{-n} y) \leq \lambda^n d(x, y)$ for $y \in W^u_\epsilon(x)$, $n \geq 0$.

(c) $W^s_\epsilon(x)$, $W^u_\epsilon(x)$ vary continuously with x in C^r topology.

<u>Corollary</u>: For $x \in N \subseteq M$ hyperbolic

$$W^s_\epsilon(x) \subset W^s(x) \ \& \ W^s(x) = \bigcup_{n \geq 0} f^{-n} W^s_\epsilon(f^n(x))$$

$$W^u_\epsilon(x) \subset W^u(x) \ \& \ W^u(x) = \bigcup_{n \geq 0} f^n W^u_\epsilon(f^{-n}(x))$$

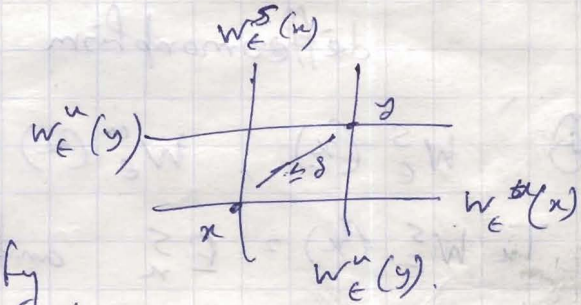
Canonical Coordinates

Let, f satisfies Axiom A. For any small $\epsilon > 0$ $\exists$ a $\delta > 0$ s.t. $W^s_\epsilon(x) \cap W^u_\epsilon(y)$ consists of a single point z, which we denote by $[x, y]$, whenever $d(x,y) <$ and, $x, y \in \Omega(f)$.

Furthermore, $[x, y] \in \Omega(f)$ and

$$[\cdot,\cdot] : \{\Omega(f) \times \Omega(f) \mid d(x,y) \leq \delta\} \longrightarrow \Omega(f) \text{ is}$$

continuous.



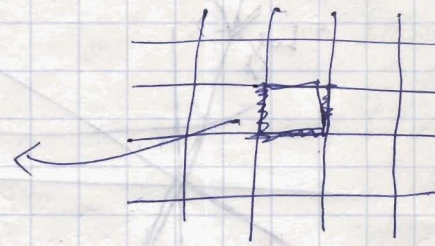
Proof sketch: $W^u_\epsilon(x) \cap W^s_\epsilon(x) = \{x\}$ is because of transversility which remains true even after δ- perturbation. $[x, y] \in \Omega(f)$ is a result of the fact that periodic points are dense in $\Omega(f)$.

Cor ~~Proof~~: Let $N \subseteq M$ hyperbolic. Then $\exists\, \epsilon > 0$ s.t. N is expansive in M, i.e. if $x \in N$ and $x \neq y \in M$, then $d(f^k x, f^k y) > \epsilon$ for some $k \in \mathbb{Z}$.

Proof: Otherwise, $y \in W^s_\epsilon(x) \cap W^u_\epsilon(x)$ and by previous theorem, $x = y$, contradiction.

~~Let~~ Examples.

1) Two-dim torus : Let, $M = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$



$\phi : \mathbb{T}^2 \longrightarrow \mathbb{T}^2$ is a map.

$(s,t) \longmapsto (s+t, s) \pmod 1$ and it has a cont. inverse $\phi^{-1} : \mathbb{T}^2 \longrightarrow \mathbb{T}^2$

$(s,t) \longmapsto (t, s-t) \pmod 1$

So, $(\mathbb{T}^2, \phi)$ is an invertible dynamical system. In particular, ϕ is an (group) automorphism of $\mathbb{R}^2/\mathbb{Z}^2$ and is called a ~~hyperbolic~~ toral automorphism.

Let, $q : \mathbb{R}^2 \longrightarrow \mathbb{T}^2$ is the projection map
$(s,t) \longmapsto (s,t) + \mathbb{Z}^2$

Let, $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ is the lin. trans. on $\mathbb{R}^2$.

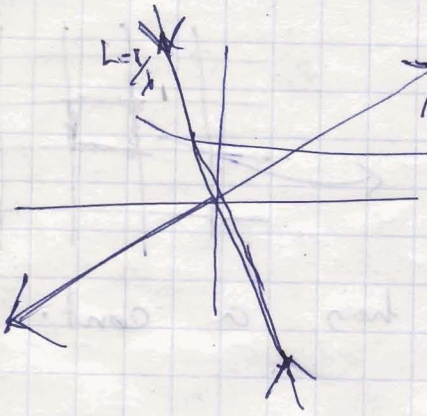
$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "A"] \arrow[d, "q"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{R}^2 \arrow[d, "q"] \\ \mathbb{T}^2 \arrow[r, "\phi"'] & \mathbb{T}^2 \end{tikzcd}$$

$A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x \end{bmatrix}$

So ϕ is the defined toral automorphism.

Note that eigenvalues of A are $\frac{1 \pm \sqrt{5}}{2}$. The Perron-Frobenius eigenvalue $\lambda = \frac{1+\sqrt{5}}{2} > 1$ and other one is $-\frac{1}{\lambda} > -1$. The eigenspaces L_λ and $L_{-1/\lambda}$ are one dimensional, ~~say L_1 and L_2~~

So, A expands in the direction (as they are one-dim direction makes sense) of L_λ and contracts and flips in the direction of $L_{-1/\lambda}$



$L_\lambda \longrightarrow$ expanding segment

$\longrightarrow$ contracting segment.

There are similar phenomena in higher dimensional tori. Let, ϕ be a toral automorphism on $\mathbb{T}^n$

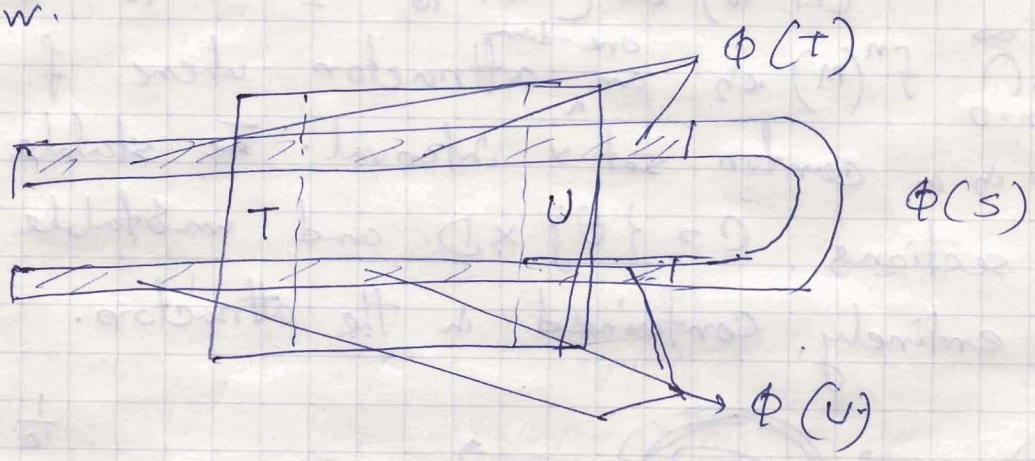
$$\phi : \mathbb{R}^n/\mathbb{Z}^n \longrightarrow \mathbb{R}^n/\mathbb{Z}^n$$
$$x \longmapsto Ax + \mathbb{Z}^n$$

where A is ~~such~~ a linear transform on $\mathbb{R}^n$, s.t. ϕ is invertible. A toral automorphism is called <u>hyperbolic</u> if no eigenvalue of A has absolute value one. Similarly there are s eigenvalues where A expands and $n-s$ where A contracts.

<u>~~Fact~~ Remark</u>: There exists orbifolds (i.e. manifold/discrete) ~~where~~ which are not hyperbolic, i.e. exists neutral direction. example, $GL_n(\mathbb{R})/GL_n(\mathbb{Z})$ has $(n-1)$ dimension of <u>flats (neutral directions)</u>.

2) <u>Horseshoe</u>

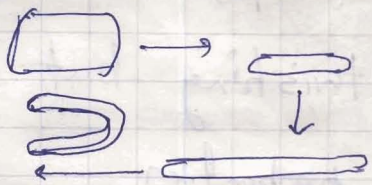
Let S be the unit square in $\mathbb{R}^2$ and let ϕ be a conf. 1-1 mapping, $\phi : S \longrightarrow \mathbb{R}^2$ s.t. $\phi(S)$ looks like below.



Both restrictions $\phi|_T$ and $\phi|_U$ are linear mappings which contract in the vertical direction and expand in the horizontal direction. ϕ is called the <u>Horseshoe mapping</u>.

(S, ϕ) is not a dynamical system since $\phi(S) \not\subset S$ we restrict ϕ to $M = \bigcap_{n=-\infty}^{\infty} \phi^n(S)$

<u>Fact</u>: ϕ is an Axiom A diffeomorphism, where stable and unstable manifolds intersect, for periodic points.



Smale's horseshoe.

M is a product of two cantor sets and is maximal invariant set for ϕ.

Here under Euclidean metric, $\lambda = \frac{1}{2}$ and $\mu = 2$ are hyperbolic splitting for vertical and horizontal direction.

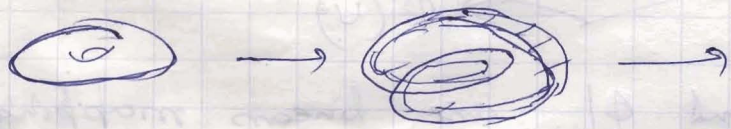
3) <u>Solenoid</u>

Let $M = S^1 \times D$ be the solid torus. Let's define

$$f : M \longrightarrow M$$

$$(\phi, x, y) \mapsto \left(2\phi,\ \tfrac{1}{10}x + \tfrac{1}{2}\cos\phi + \tfrac{1}{10}y + \tfrac{1}{2}\sin\phi\right)$$

Then $M_0 = \bigcap_{n=0}^{\infty} f^n(M)$ is an one-dim attractor where f is expanding locally it is a cantor set $\times$ interval. The stable manifolds are the sections $C = \{\theta\} \times D$ and unstable manifolds are entirely contained in the attractor.



$\frac{1}{10} < \frac{1}{2}$

$\frac{1}{2} + \frac{1}{10} < 1$

The map f is a smooth embedding of T onto itself the preserves the foliation by (meridional disk) cross-sectional disks. The map f expands in the longitudinal direction, contracts cross-sectional disk and then wraps the deformed ~~torus~~ tube twice inside T with twisting but without self intersection.

<u>properties</u>

(i) $\{\theta\} \times D$ i.e. cross-sectional disks are stable mfd.

(ii) & they are cantor set in M_0.

(iii) periodic points of f is dense

} f is axiom A.

(iv) f is topologically transitive.

Let f be an Axiom A diffeomorphism. $\Omega(f)$ is the set non-wandering points of f. Then,

<u>Theorem</u>: $\Omega(f) = \bigcup_{i=1}^{s} \Omega_i$, $\Omega_i \cap \Omega_j = \phi$, $i \neq j$, closed.

(i) $f(\Omega_i) = \Omega_i$ and $f|_{\Omega_i}$ is topo-trans.

(ii) $\Omega_i = \bigcup_{j=1}^{n_i} X_{j,i}$, $X_{j,i} \cap X_{k,i} = \phi$, $j \neq k$, closed

$f(X_{j,i}) = X_{j+1,i \pmod{n_i}}$ and $f^{n_i}|_{X_{j,i}}$ is topologically mixing.

This theorem is called "Spectral decomposition".

It was a long-standing question if f is Anosov whether $\Omega(f) = M$ or not. (False).

<u>Anosov closing lemma</u>: If f is Anosov diff. (M is hyp), then f satisfies Axiom A.

~~Proof~~: <u>Fundamental Neighbourhood</u>

If f satisfies axiom A, $\exists$ a nbd U of $\Omega(f)$ s.t. $\bigcap_{n \in \mathbb{Z}} f^n(U) = \Omega(f)$

$W^s(x) = \{ x \cdot n(a) \mid a \in \mathbb{R} \}$, $n(a) = \begin{pmatrix} 1 & a \\ & 1 \end{pmatrix}$

$W^u(x) = \{ x \cdot n'(b) \mid b \in \mathbb{R} \}$, $n'(b) = \begin{pmatrix} 1 & \\ b & 1 \end{pmatrix}$.

$T\mathcal{H}^2 = E^+ \oplus E^0 \oplus E^-$

A <u>flow</u> $\phi : M \times \mathbb{R} \longrightarrow M$.

$\phi(x, 0) \longmapsto x$.

$\phi(\phi(x,t), s) = \phi(x, t+s)$.

An <u>Anosov flow</u> $g_t : TM \longrightarrow TM$.

(1) <u>transversal</u>: $T_x M = T_x(g_{\mathbb{R}} x) \oplus T_x W^s(x) \oplus T_x W^u(x)$

(2) $g_t W^s(x) = W^s(g_t x)$ <u>stability</u>

$g_t W^u(x) = W^u(g_t x)$

(3) <u>Contraction</u>: $y \in W^s(x)$

$d(g_t x, g_t y) \leq c e^{-t} d(x, y)$

$y \in W^u(x)$

$d(g_t x, g_t y) \leq c e^{-t} d(x, y)$.

<u>Anosove closing lemma</u>

Let. For some $x \in X$, $t > 0$, $d(x, g_t x) < \epsilon$ (ϵ given)

then, $\exists$ periodic $x_0 \in X$ s.t. $d(x, x_0) < \epsilon$