• <u>Phase transition for Ising model on $\mathbb{Z}^2$</u>. [Peierls argument]

References: [C], [GHM]. $\boxed{1}$

• Two constructions for measures μ_β^+, μ_β^-. $\beta = T^{-1}$ = inverse temp.,
as Ben described (and in class).
We will show that there are two regimes: high vs. low temperature.

<u>High temperatures</u>: entropy dominates energy.
The interaction between constituent particles is not significant enough to affect macroscopic behavior of system, so long-range interactions become negligible.
So, the center of a large box is largely unaffected by boundary conditions.

<u>Low temperatures</u>: residual effects of boundary conditions are felt. eg, imagine taking lattice with forced magnetization on boundary of larger and larger sets — even as the distance between boundary and center of the lattice goes $\to \infty$, residual effect will be seen at the center.

<u>Setup</u>: $\Lambda \subset \mathbb{Z}^2$ finite, containing origin. $|\Lambda| = N$.
Configuration σ on Λ determines spins, ± 1, at each site.
Denote $\sigma = (\sigma_1, \ldots, \sigma_N) \in \{\pm 1\}^\Lambda$.
Hamiltonian: $H(\sigma) = -\sum_{(i,j)} E\sigma_i\sigma_j$ (no external field)

Probability of configuration $\sigma \in \Omega = \{$configs on Λ st $\sigma|_{\partial\Lambda} \equiv +1\}$:
$\mathbb{P} := \mu_\beta^{+,\Lambda}$; $Z = \sum_{\sigma\in\Omega} e^{-\beta H(\sigma)}$ = partition function;

$\mathbb{P}(\sigma) = \frac{1}{Z} e^{-\beta H(\sigma)}$.

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Label lattice so that $\sigma_0 \leftrightarrow$ spin at origin.
Then, if we denote

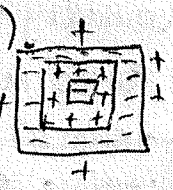
$$\Omega_0 = \{\text{configs } \sigma \text{ st } \sigma_0 = -1\} \subset \Omega,$$

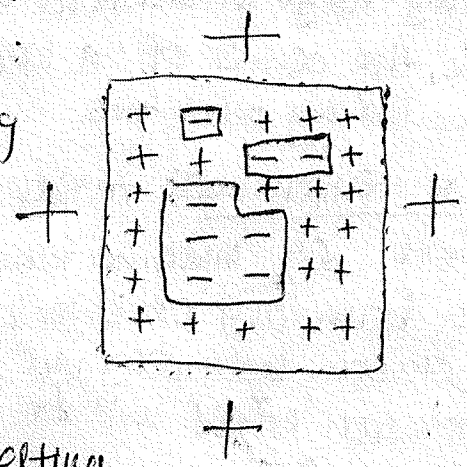
$$\mathbb{P}(\sigma_0 = -1) = \frac{1}{Z} \sum_{\sigma \in \Omega_0} e^{-\beta H(\sigma)} \qquad (1)$$

We will show (1) is strictly less than $\frac{1}{2}$, independently of the size of the lattice $|\Lambda|$.

Typical configuration for $\sigma \in \Omega_0$:

Think of ocean of (+) containing islands of (−).

(Possible that  ; no worries)



Every island is surrounded by a <u>shoreline</u> := closed path connecting midpoints of adjacent lattice sites;

every segment (i,j) of a shoreline $\longleftrightarrow \sigma_i \sigma_j = -1$, ie, separates (+) from (−).

Since $\sigma_0 = -1 \ \forall \sigma \in \Omega_0$, $\exists$ shoreline S around 0.

Call length of $S \leftrightarrow n(S)$.

Call $\Omega_S := \{\sigma \in \Omega_0 : S \text{ is a shoreline for } \sigma\}$.

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$$\mathbb{P}(\Omega_S) = \frac{1}{Z} \sum_{\sigma \in \Omega_S} e^{-\beta H(\sigma)}$$

$\sigma_i \sigma_j = -1$ for $(i,j) \in S$.

$$= \frac{1}{Z} \sum_{\sigma \in \Omega_S} e^{-\beta \left[-\sum_{(i,j)\in S} \sigma_i \sigma_j - \sum_{(i,j)\notin S} \sigma_i \sigma_j \right]}$$

$$= \frac{1}{Z} \sum_{\sigma \in \Omega_S} e^{-\beta E n(S)} \cdot e^{\beta E \sum_{(i,j)\notin S} \sigma_i \sigma_j}$$

$$= \frac{e^{-\beta E n(S)}}{Z} \sum_{\sigma \in \Omega_S} e^{\beta E \sum_{(i,j)\notin S} \sigma_i \sigma_j} \qquad (2)$$

Now: for $\sigma \in \Omega_S$, can form another config. $\sigma' \in \Omega$ by changing (flipping) all spins inside shoreline S.
For fixed S, $\sigma \mapsto \sigma'$ is injective. Call $\Omega_S' =$ image of Ω_S under this.
Clear that for $(i,j) \notin S$, $\sigma_i \sigma_j = \sigma_i' \sigma_j'$,
and $\sigma_i \sigma_j = -1$ for $(i,j) \in S \Rightarrow \sigma_i' \sigma_j' = +1$ for $(i,j) \in S$.

$$\Rightarrow \sum_{(i,j)\notin S} \sigma_i \sigma_j = \sum_{(i,j)\notin S} \sigma_i' \sigma_j' = \sum_{(i,j)} \sigma_i' \sigma_j' - \sum_{(i,j)\in S} \sigma_i' \sigma_j'$$

$$= \sum_{(i,j)} \sigma_i' \sigma_j' - n(S).$$

So, $\sum_{(i,j)\notin S} \sigma_i \sigma_j < \sum_{(i,j)} \sigma_i' \sigma_j'$.

Plug into (2):

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$$\mathbb{P}(\Omega_S) < \frac{e^{-\beta E n(S)}}{Z} \sum_{\sigma \in \Omega_S} e^{\beta E \sum_{(i,j)} \sigma_i' \sigma_j'}$$

$$\sigma \mapsto \sigma' \text{ 1-1} \Rightarrow \quad = \frac{e^{-\beta E n(S)}}{Z} \sum_{\sigma' \in \Omega_S'} e^{\beta E \sum_{(i,j)} \sigma_i' \sigma_j'}$$

$$= \frac{e^{-\beta E n(S)}}{Z} \sum_{\sigma' \in \Omega_S'} e^{-\beta H(\sigma')}$$

$$\Omega_S \subset \Omega \quad \leq \frac{e^{-\beta E n(S)}}{Z} \sum_{\sigma \in \Omega} e^{-\beta H(\sigma)}$$

$$= e^{-\beta E n(S)} \qquad (3)$$

Let $\mathcal{S}$ = set of shorelines surrounding 0 in Λ.

$$\mathbb{P}(\sigma_0 = -1) = \sum_{S \in \mathcal{S}} \mathbb{P}(\Omega_S)$$

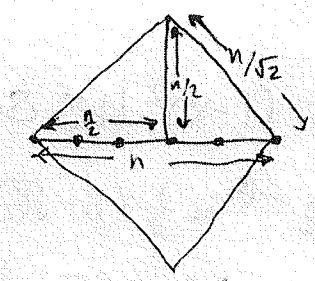
$$< \sum_{S \in \mathcal{S}} e^{-\beta E n(S)}$$

$$= \sum_{n=4}^{\infty} s(n) e^{-\beta E n},$$

$s(n)$ = # shorelines of length n surrounding 0.

So, suffices to bound $s(n)$.

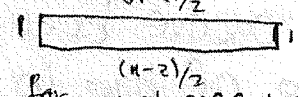
A shoreline of length n is contained in a square of side length $\leq n/\sqrt{2}$.



Let

$r(n) = \#$ paths of a random walk of length n originating in a square of side $n/\sqrt{2}$.

(Actually need



for worst case, but side length is still order n)

Every shoreline is closed path of length n; so every starting segment corresponds to a random walk of that length.

So $\quad n \cdot s(n) < r(n) \Rightarrow s(n) < \frac{1}{n} r(n)$.

Bound $r(n)$: $\left(\frac{n}{\sqrt{2}}\right)^2 = n^2/2$ possible starting positions inside square. At most 4^n possible paths. ($< 4 \cdot 3^{n-1}$, if use non-backtrack; no matter, though)

$$\Rightarrow \quad s(n) < \frac{1}{2} \cdot n \cdot 4^n.$$

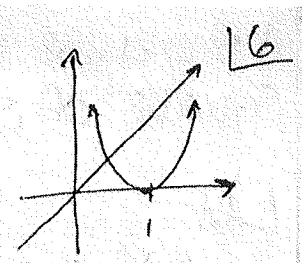
$$\Rightarrow \mathbb{P}(\sigma_0 = -1) < \sum_{n=4}^{\infty} \frac{1}{2} \cdot n \cdot 4^n \cdot e^{-\beta n E}$$

$$< \frac{1}{2} \sum_{n=1}^{\infty} n \cdot (4 e^{-\beta E})^n.$$

$$\frac{1}{1-x} = 1 + x + \dots = \sum_{0}^{\infty} x^n, \quad |x| < 1.$$

$$\Rightarrow \frac{x}{1-x} = \sum_{n=0}^{\infty} x^{n+1} \Rightarrow \frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} n x^n, \quad |x| < 1.$$

$$\Rightarrow \mathbb{P}(\sigma_0 = -1) < \frac{1}{2}\left[\frac{4e^{-\beta E}}{(1-4e^{-\beta E})^2}\right]$$



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$x = (1-x)^2$ has solution $\frac{1}{2}(3 \pm \sqrt{5})$.

$4e^{-\beta} = \frac{1}{2}(3-\sqrt{5}) \Rightarrow \beta \approx 2.35$

So, for $\beta > 2.35$, $\mathbb{P}(\sigma_0 = -1) < \frac{1}{2}$ $(E=1)$

The above bound holds independently of the size of the lattice. So, holds for $\mu_\beta^+ = \lim_{\Lambda \uparrow \mathbb{Z}^d} \mu_\beta^{+,\Lambda}$.

Symmetry shows that for $\mu_\beta^{-,\Lambda}(\sigma_0 = +1) < \frac{1}{2}$,

hence $\mu_\beta^- \neq \mu_\beta^+$ for $\beta > 2.35$.

<u>Known</u>: $\beta_c = \frac{1}{2}\log(1+\sqrt{2})$ by Onsager, studying singularities
≈ 0.441

of "free energy per lattice site", $F(\beta, E, h) = \lim_{N\to\infty} \frac{1}{N} \log Z(\beta, E, h, N)$.

For $\beta < \beta_c$, $\mu^+ = \mu^-$.

[GHM] showed, by coupling argument, that if ψ_p = measure for Bernoulli site percolation on $\mathbb{Z}^2$, then for any $\Delta \subset \Lambda$,

$$\|\mu_\Lambda^+ - \mu_\Lambda^-\|_\Delta \leq \psi_p(\Delta \leftrightarrow \partial\Lambda),$$

where $\|\nu\|_\Delta = \sup_{A \in \mathcal{F}_\Delta} |\nu(A)|$ = total variation norm on $\mathcal{F}_\Delta$ = {σ-alg. of events depending only on spins in Δ},

$\psi_{\bar{p}}(\sigma_x = 1) = p_x$

and $\bar{p} = (p_x) = \max_{\eta,\eta' \in \Omega} \|\mu_x^\eta - \mu_x^{\eta'}\|_x$, η, η': boundary cond'ns.

Let $\Lambda \uparrow \mathbb{Z}^2 \Rightarrow \|\mu^+ - \mu^-\|_\Delta \leq \psi_p(\Delta \leftrightarrow \infty)$

By spin-flip symmetry & stochastic monotonicity, 7

$$\mu^+_{\beta,\Lambda}(\sigma_x) = \mu^+_{\beta,\Lambda}(\sigma_x = 1) - \mu^+_{\beta,\Lambda}(\sigma_x = -1)$$

$$= \mu^+_{\beta,\Lambda}(\sigma_x = 1) - \mu^-_{\beta,\Lambda}(\sigma_x = 1)$$

$$= \| \mu^+_{\beta,\Lambda} - \mu^-_{\beta,\Lambda} \|_x .$$

$$\begin{array}{ccc} & + & \\ + & \oplus & + \\ & + & \end{array} \qquad \begin{array}{ccc} & - & \\ - & \oplus & - \\ & - & \end{array}$$

$H = -4$ $\qquad$ $H = 4$

$$\mu^+_{\beta,x}(\sigma_x = 1) = \frac{e^{4\beta}}{Z}, \quad \mu^-_{\beta,x}(\sigma_x = 1) = \frac{e^{-4\beta}}{Z},$$

so $$\| \mu^+_{\beta,x} - \mu^-_{\beta,x} \| = \frac{e^{4\beta} - e^{-4\beta}}{e^{4\beta} + e^{-4\beta}} = \tanh(4\beta).$$

So, enough to check for singletons; if $\tanh(4\beta) < p_c$ for site percolation in $\mathbb{Z}^2$, know $\exists!$ Gibbs.

Currently known $0.556 < p_c < 0.680$.

So this shows $\beta \lesssim 0.157$ has unique Gibbs.