Phase transition for Ising model on \( \mathbb{Z}^2 \), [Pierls argument].

References: [C], [GHM].

Two constructions for measures \( \mu^+_\beta, \mu^-_\beta \), \( \beta = T^{-1} \) inverse temp., as Ben described (and in class).

We will show that there are two regimes: high vs. low temperature.

High temperatures: entropy dominates energy. The interaction between constituent particles is not significant enough to affect macroscopic behaviour of system, so long-range interactions become negligible. So, the center of a large box is largely unaffected by boundary conditions.

Low temperatures: residual effects of boundary conditions are felt. E.g., imagine taking lattice with forced magnetization on boundary of larger and larger sets — even as the distance between boundary and center of the lattice goes \( \to \infty \), residual effect will be seen at the center.

Setup: \( \Lambda \subset \mathbb{Z}^2 \) finite, containing origin. \( |\Lambda| = N \).

Configuration \( \sigma \) on \( \Lambda \) determines spins, \( \pm 1 \), at each site. Denote \( \sigma = (\sigma_1, \ldots, \sigma_N) \in \mathcal{S}_\Lambda = \{ \pm 1 \}^\Lambda \).

Hamiltonian: \( H(\sigma) = -\sum_{<i,j>} \sigma_i \sigma_j \) (no external field).

Probability of configuration \( \sigma \in \mathcal{S}_\Lambda \), \( \sigma \) configs on \( \Lambda \) s.t. \( \sigma|_{\partial\Lambda} = +1 \):

\[ P = \mu^+_\beta; \quad Z = \sum_{\sigma \in \mathcal{S}_\Lambda} e^{-\beta H(\sigma)} = \text{partition function} \]
\[ P(\sigma) = \frac{1}{Z} e^{-\beta H(\sigma)} \]

Label lattice so that \( \sigma_0 \leftrightarrow \) spin at origin.
Then, if we denote
\[ \Omega_0 = \{ \text{config} \ \sigma \text{ st } \sigma_0 = -1 \} \subset \Omega, \]
\[ P(\sigma_0 = -1) = \frac{1}{Z} \sum_{\sigma \in \Omega_0} e^{-\beta H(\sigma)} \quad (1) \]

We will show (1) is strictly less than \( \frac{1}{Z} \), independently of the size of the lattice \( |\Lambda| \).

Typical configuration for \( \sigma \in \Omega_0 \):
Think of ocean of (+) containing islands of (-).
(Possible that + \[ \begin{array}{c} + \\ + \end{array} \] ; no worries)

Every island is surrounded by a **shoreline** := closed path connecting midpoints of adjacent lattice sites;
every segment \((i,j)\) of a shoreline \( \leftrightarrow \sigma_i \sigma_j = -1, \)
ie, separates (+) from (-).
Since \( \sigma_0 = -1 \ \forall \ \sigma \in \Omega_0, \) \( \exists \) shoreline \( S \) around 0.
Call length of \( S \leftrightarrow n(S) \).

Call \( \Omega_S = \{ \sigma \in \Omega_0 : S \text{ is a shoreline for } \sigma \} \).
\[ P(\Omega_S) = \frac{1}{Z} \sum_{\sigma \in \Omega_S} e^{-\beta H(\sigma)} \]

\[ = \frac{1}{Z} \sum_{\sigma \in \Omega_S} e^{-\beta \sum_{(i,j) \in S} \sigma_i \sigma_j - \sum_{(i,j) \notin S} \sigma_i \sigma_j} \]

\[ = \frac{1}{Z} \sum_{\sigma \in \Omega_S} e^{-\beta E_\sigma(S)} \cdot e^{\beta \sum_{(i,j) \notin S} \delta_i \delta_j} \]

\[ = \frac{e^{-\beta E_\sigma(S)}}{Z} \sum_{\sigma \in \Omega_S} e^{\beta \sum_{(i,j) \notin S} \delta_i \delta_j} \] 

(2)

Now, for \( \sigma \in \Omega_S \), can form another config. \( \sigma' \in \Omega \)
by changing (flipping) all spins inside subregion \( S \).
For fixed \( S \), \( \sigma \mapsto \sigma' \) is injective. Call \( \Omega_S' \) = image of \( \Omega_S \) under this.
Clear that for \( (i,j) \notin S \), \( \sigma_i \sigma_j = \sigma'_i \sigma'_j \),
and \( \sigma_i \sigma_j = -1 \) for \( (i,j) \in S \) \( \Rightarrow \sigma'_i \sigma'_j = +1 \) for \( (i,j) \in S \).

\[ \sum_{(i,j) \notin S} \sigma_i \sigma_j = \sum_{(i,j) \notin S} \sigma'_i \sigma'_j = \sum_{(i,j) \notin S} \sigma_i \sigma_j' - \sum_{(i,j) \notin S} \sigma_i \sigma_j' = \sum_{(i,j) \notin S} \delta_i \delta_j' - \langle S \rangle. \]

So, \( \sum_{(i,j) \notin S} \sigma_i \sigma_j' < \sum_{(i,j) \notin S} \sigma_i \sigma_j' \).

Plug into (2).
\[ P(\mathcal{S}_S) < e^{-\beta E(n)} \leq e^{\beta E(\sigma)} \leq e^{\beta \underline{E}(\sigma')} \]

\[ 0 \rightarrow \sigma' \quad 1 \rightarrow \sigma \]

\[ = \frac{e^{-\beta E(n)}}{Z} \leq e^{\beta \underline{E}(\sigma')} \]

\[ = \frac{e^{-\beta E(n)}}{Z} \leq e^{-\beta H(\sigma')} \]

\[ \mathcal{S}_S < \mathcal{S} \quad \leq \frac{e^{-\beta E(n)}}{Z} \leq e^{-\beta H(\sigma)} \]

\[ = e^{-\beta E(n)} \quad (3) \]

Let \( \mathcal{S} \) = set of shorelines surrounding 0 in \( \Lambda \).

\[ P(\sigma_0 = -1) = \frac{\sum_{\mathcal{S}} P(\mathcal{S}_S)}{\mathcal{S} \in \mathcal{S}} \leq e^{-\beta E(n)} \]

\[ \leq \frac{e^{-\beta E(n)}}{Z} \leq s(n) e^{-\beta E(n)} \]

\[ s(n) = \# \text{shorelines of length } n \text{ surrounding } 0 \]

So, suffices to bound \( s(n) \).
A shoreline of length \( n \) is contained in a square of side length \( \frac{n}{\sqrt{2}} \).

Let

\[
\begin{align*}
\mathbf{r}(n) &= \text{# paths of a random walk of length } n \text{ originating in a square of side } \frac{n}{\sqrt{2}}. \\
\end{align*}
\]

Every shoreline is closed path of length \( n \); so every starting segment corresponds to a random walk of that length.

\[
\begin{align*}
\Rightarrow \quad n \cdot s(n) \leq r(n) \Rightarrow s(n) &< \frac{1}{n} r(n). \\
\end{align*}
\]

Bound \( r(n) \): \( \left( \frac{n}{\sqrt{2}} \right)^2 = \frac{n^2}{2} \) possible starting positions inside square. At most \( 4^n \) possible paths. \( \left( \leq 4 \cdot 3^{n-1} \right. \\
\text{use non-backtrack, no matter how) }
\]

\[
\begin{align*}
\Rightarrow s(n) &< \frac{1}{2} \cdot n \cdot 4^n. \\
\end{align*}
\]

\[
\begin{align*}
\Rightarrow p(\sigma = -1) &< \frac{1}{2} \cdot n \cdot 4^n \cdot e^{-\beta n E} \\
&< \frac{1}{2} \cdot \frac{58}{n} \cdot (4e^{-\beta E})^n. \\
\end{align*}
\]

\[
\begin{align*}
\frac{1}{1-x} &= 1 + x + \ldots \Rightarrow 0 < x^n, |x| < 1. \\
\Rightarrow \quad \frac{x}{(1-x)^2} &= \sum_{n=0}^{\infty} n x^n \quad |x| < 1.
\end{align*}
\]
\[ P(\sigma_0 = -1) < \frac{1}{2} \left[ \frac{4 e^{-\beta E}}{(1 - 4 e^{-\beta E})^2} \right] \]

\[ \chi = (1 - \chi)^2 \text{ has solution } \frac{1}{2} \left( 3 \pm \sqrt{5} \right) . \]

\[ 4 e^{-\beta} = \frac{1}{2} (3 - \sqrt{5}) \Rightarrow \beta \approx 2.35 \]

\( \text{So, for } \beta > 2.35, \ P(\sigma_0 = -1) < \frac{1}{2} \quad (\epsilon = 1) \)

The above bound holds independently of the size of the lattice. So, holds for \( \mu_\beta^+ = \lim_{\Lambda \to \infty} \mu_\beta^+ \Lambda \)

Symmetry shows that for \( \mu_\beta^-(\sigma_0 = +1) < \frac{1}{2} \), hence \( \mu_\beta^- \neq \mu_\beta^+ \) for \( \beta > 2.35 \).

**Known:** \( \beta_c = \frac{1}{2} \log (1 + \sqrt{2}) \) by Onsager, studying singularities \( \approx 0.441 \)

of "free energy per lattice site", \( F(\beta, E, h) = \lim_{N \to \infty} \frac{1}{N} \log Z(\beta, E, h, N) \).

For \( \beta < \beta_c, \ \mu^+ = \mu^- \).

[GHM] showed, by coupling argument, that if \( \Psi_\beta = \text{measure for Bernoulli site percolation on } \mathbb{Z}^2 \), then for any \( \Delta \subset \Lambda \),

\[ ||\mu_\beta^+ - \mu_\beta^-||_\Delta \leq \Psi_\beta(\Delta \subset \partial \Lambda) , \]

where \( ||\cdot||_\Delta = \sup_{A \in \mathcal{F}_\Delta} |\cdot(A)| \) = total variation norm on \( \mathcal{F}_\Delta = \sigma\text{-alg. of events depending only on spins in } \Delta \).

\[ \Psi_\beta(\sigma_x = 1) = \mathbb{P}_x \]

and \( \mathbb{P}(P) = \max_{\gamma \in \partial \mathbb{Z}^2} \| \mu_\gamma^0 - \mu_\gamma^+ \|_{\chi} , \gamma|\text{ boundary conditions} \)

Let \( \Lambda \subset \mathbb{Z}^2 \Rightarrow ||\mu - \mu^-|| \leq \Psi_\beta(\Lambda \subset \partial \infty) \)
By spin-flip symmetry & stochastic monotonicity,
\[ \mu_{\beta,x}^+(\sigma_x = 1) - \mu_{\beta,x}^- (\sigma_x = 1) \]
\[ = \mu_{\beta,x}^+ (\sigma_x = 1) - \mu_{\beta,x}^- (\sigma_x = 1) \]
\[ = \| \nu_{\beta,x}^+ - \nu_{\beta,x}^- \|_{X}. \]

So, enough to check for singletons; if
\[ \tanh(4\beta) < p_c \] for site percolation in \( \mathbb{Z}^2 \), know 3! Gibbs

Currently known \( 0.556 < p_c < 0.680 \).

So this shows \( p \leq 0.157 \) has unique Gibbs.