

Markov Partitions for Hyperbolic Maps

— Basic Concepts and examples.

I. Motivation

There is no wonder to analyse general discrete time dynamical systems by symbolic dynamical systems (e.g. shifts of finite type). However, representing a general dynamical system by a symbolic one involves fundamental complication.

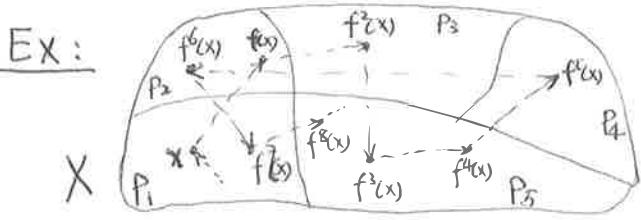
In general, we want, for a general D.S. (X, f) ,

- ① a continuous, 1-1 correspondence between the orbits $\{f^n(x)\}$ and the orbits $\{\sigma^n(a_k)\}$, where $x \in (X, f)$ and $a_k \in (\Sigma_A, \sigma)$ — a shift space with alphabet A ;
- ② σ to be shift of finite type.

But ① and ② are usually in conflict:

- ① implies that X is homeomorphic to Σ_A . On the other hand, X is often a smooth manifold, while Σ_A is ~~discrete~~ totally disconnected.

Therefore, partitioning is brought into play.



Partition X into disjoint opens $\{P_1, P_2, P_3, P_4, P_5\}$
 $\vdash P$, s.t. $\overline{P_1} \cup \overline{P_2} \cup \overline{P_3} \cup \overline{P_4} \cup \overline{P_5} = X$.
representation of $x \in (X, f)$; ... 1 2 3 5 5 4 2 1 5 ...
by tracking in which P_i , $i=1, 2, \dots, 5$, $f^n(x)$ lands, $n \in \mathbb{Z}$.

Let Σ_P be the set of all representations, $\Sigma_P \subseteq \{1, 2, 3, 4, 5\}^{\mathbb{Z}}$, σ ~~be~~ a shift.

The idea is to understand (X, f) using (Σ_P, σ) .

So P must be well chosen, in the sense that there will be only one point $x \in X$ corresponding to each representation $a_k \in \Sigma_P$. This means that

$\varphi: \Sigma_P \rightarrow X$ is onto, but not necessarily 1-1.

In the previous example,

$$\begin{array}{ll} f^{-1}(x) \in P_{\alpha_1} & x \in f(P_{\alpha_1}) \\ x \in P_{\alpha_0} & \Leftrightarrow x \in P_{\alpha_0} \\ f(x) \in P_{\alpha_1} & x \in f^{-1}(P_{\alpha_1}) \Leftrightarrow x \in \bigcap_{n \in \mathbb{Z}} f^{-n}(P_{\alpha_1}). \\ f^2(x) \in P_{\alpha_2} & x \in f^{-2}(P_{\alpha_2}) \end{array}$$

We want $\bigcap_{n \in \mathbb{Z}} f^{-n}(P_{\alpha_1}) = \{x\}$.

II. Definitions and Properties

Def: Let (X, f) be a general dynamical system, invertible. A topological partition $P = \{P_0, P_1, \dots, P_{r-1}\}$ of X gives a symbolic representation of (X, f) if for every $a \in \Sigma_P$ the intersection $\bigcap_{n \geq 0} \overline{D_n}(a)$ consists of exactly one point, where a topological partition is a finite collection $P = \{P_0, P_1, \dots, P_{r-1}\}$ of disjoint open sets st. $X = \bigcup_{j=0}^{r-1} \overline{P_j}$, and $D_n(a) = \bigcap_{k=-n}^n f^{-k}(P_{a_k}) \subseteq X$.

There is a one-sided version of symbolic representation for non-invertible dynamical systems, replacing Σ_P by Σ_P^+ and $D_n(a)$ by $D_n^+(a) = \bigcap_{k \geq 0} f^{-k}(P_{a_k})$.

The next example will explain why we choose $\overline{D_n}(a)$ and why $\varphi: \Sigma_P \rightarrow X$ is not necessarily 1-1.

Ex: Let $X = \mathbb{T}$ be the circle \mathbb{R}/\mathbb{Z} . $f: x \mapsto 10x \pmod{1}$.

$P = \left\{ \left(\frac{i}{10}, \frac{i+1}{10} \right) \mid i=0, 1, \dots, 9 \right\}$. So the ~~alphabet~~ symbol set $A = \{0, 1, \dots, 9\}$

For $x \in X$ and $n \geq 0$, define a ~~symbol~~ a & $a \in A$ by $f^n(x) \in \left(\frac{a_n}{10}, \frac{a_{n+1}}{10} \right)$

Then $a = a_0 a_1 a_2 \dots \in \Sigma_P^+ = A^{\mathbb{N}}$ corresponds to the sequence of digits in its decimal expansion, since f corresponds to the (one-sided)

shift map, i.e., multiplying x by 10 shifts the decimal digits to the left by one and deletes the left-most digit since mod 1.

$$\text{e.g. } x = \pi - 3 = 0.\overline{1415926535897} \dots \xrightarrow{\quad f(x) = 0.\overline{415926535897} \dots \quad} \sigma(a) = 1\overline{415926535897} \dots$$

But $\varphi: \Sigma_p^+ \rightarrow X$ is not injective.

e.g. $0.\overline{2000} \dots = 0.\overline{1999} \dots$. Every decimal number has 2 preimages.

$\{\alpha \mid \alpha \text{ corresponds to } x\}$ is not a shift space, since it may not be compact. e.g., $1\overline{9000} \dots, 1\overline{99000} \dots, 1\overline{999000} \dots, \dots$ has limiting sequence $1\overline{999999} \dots$, which is not compact. Taking $\overline{D_\alpha(\alpha)}$ makes it $\overline{D_\alpha(\alpha)}$ compact.

Now we move to the definition of Markov partition for a hyperbolic map.

Recall: For a hyperbolic set Λ^{hyp} , $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|W_\varepsilon^s(x) \cap W_\varepsilon^u(y)| = 1$, $[x, y] := W_\varepsilon^s(x) \cap W_\varepsilon^u(y)$, whenever $x, y \in \Lambda$, and $d(x, y) < \delta$, M is a compact Riemannian manifold. Also, $[\cdot, \cdot]: \{(x, y) \in \Lambda \times \Lambda \mid d(x, y) < \delta\} \rightarrow \overset{M}{\Lambda}$ is continuous. Note that $[\cdot, \cdot]$ maps into Λ .

We introduce a particular ~~kind of~~ class of hyperbolic sets.

Def: A hyperbolic set Λ for a diffeomorphism f is said to be locally maximal if \exists open neighbourhood U of Λ s.t. $\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(\overline{U})$. Λ is also called basic.

Remark: If U is sufficiently small, then all the orbits ~~lie in~~ Λ . Any periodic point ~~in~~ in U is contained in ~~in~~ $\bigcap_{n \in \mathbb{Z}} f^n(\overline{U})$. If U is sufficiently small, then all the orbits of f are in Λ .

Fact: For a locally maximal hyperbolic set Λ , if $x, y \in \Lambda$, $d(x, y) \leq \delta$ for some $\delta > 0$, then $[x, y] \in \Lambda$.

Def: A ~~closed~~ subset $R \subset \Lambda$ is called rectangle if

1. $\text{diam } R < \delta$, for small $\delta > 0$.
2. $[x, y] \in R$ whenever $x, y \in R$.

R is called proper if R is closed and $R = \overline{\text{int}(R)}$ in the topology of Λ .

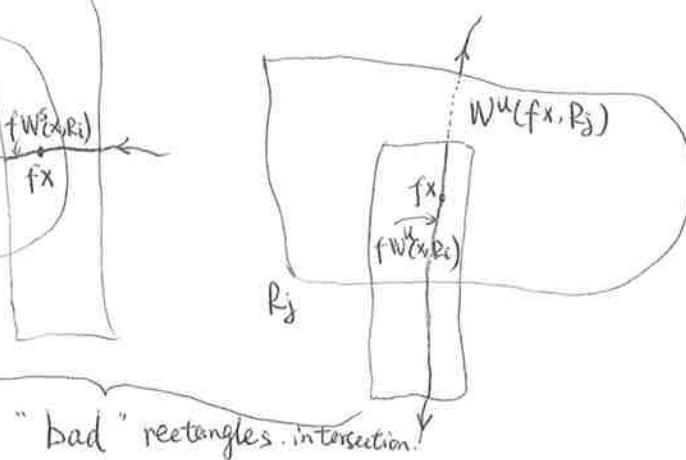
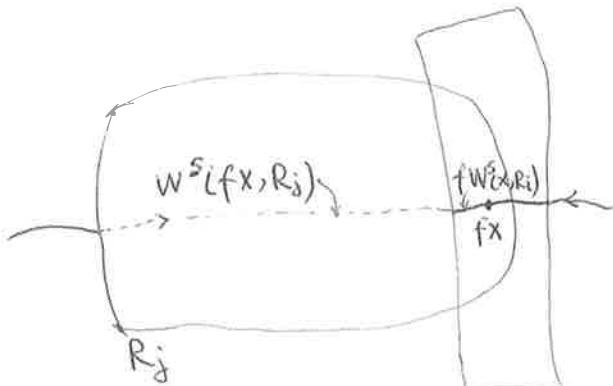
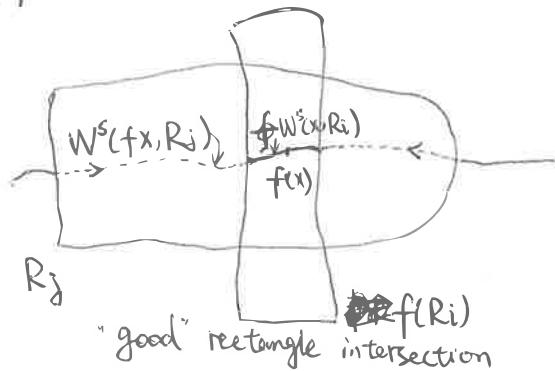
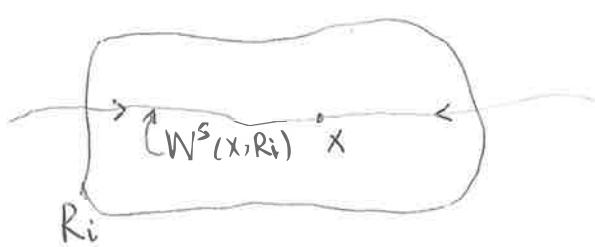
Let $W^s(x, R) = W_\epsilon^s(x) \cap R$ and $W^u(x, R) = W_\epsilon^u(x) \cap R$, $\delta < \epsilon$.

Def: A Markov partition of $\partial\Lambda$ (with respect to f) is a finite covering $\mathcal{R} = \{R_1, \dots, R_m\}$ of $\partial\Lambda$ by proper rectangles with

(a) $\text{int}(R_i) \cap \text{int}(R_j) = \emptyset$ for $i \neq j$.

(b) $f W^u(x, R_i) \supset W^u(fx, R_j)$ and

$f W^s(x, R_i) \subset W^s(fx, R_j)$, if $x \in \text{int}(R_i)$, $fx \in \text{int}(R_j)$.



Theorem 1: Any ^{compact} locally maximal hyperbolic set has Markov partitions of ~~arbitrarily~~ arbitrarily small diameter.

Theorem 2: Let the transition matrix $A = A(R)$ be defined by

$$A_{ij} = \begin{cases} 1, & \text{if } \text{int}(R_i) \cap f^{-1}\text{int}(R_j) \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

Then for each $a \in \Sigma_A$, the set $\bigcap_{j \in \mathbb{Z}} f^{-j} R_{\alpha j} = \{\pi(a)\}$.

The map $\pi: \Sigma_A \rightarrow \Lambda$ is a continuous surjection, s.t. $\pi \circ \sigma = f \circ \pi$. 

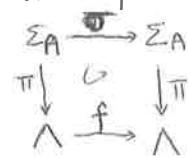
Moreover, Furthermore, π is 1-1 over the residue set

$$Y = \bigwedge \bigvee_{\substack{S \in \mathcal{P} \\ S \neq \emptyset}} f(S^s R \cup S^u R), \text{ where}$$

$$\partial^s R = \sum_j \partial^s R_j \quad \text{and} \quad \partial^{up} R = \sum_j \partial^{up} R_j;$$

$$\delta^e R_j = \{ x \in R_j \mid x \notin \text{int}(W^s(x, R_j)) \}, \quad \delta^e R_j = \{ x \in R_j \mid x \notin \text{int}(W^u(x, R_j)) \}.$$

The proof of the above theorem requires some work. We see some examples first and finish the proof if time permits. Theorem 2 tells us that Markov partition admits a symbolic representation of Λ .



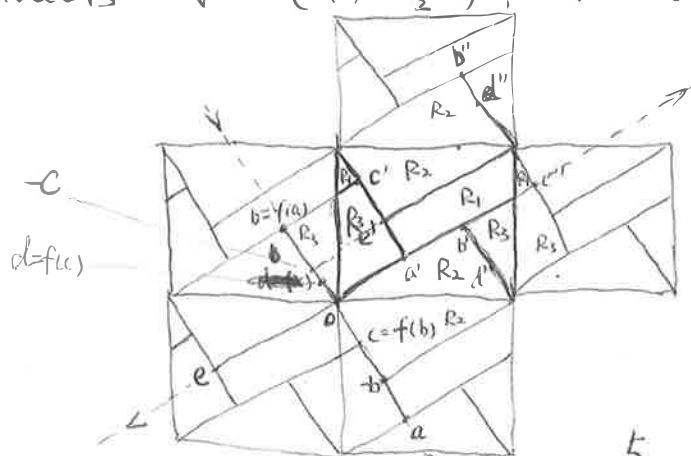
III. Examples (All of them are locally maximal hyperbolic set).

1. Hyperbolic Toral Automorphism.

$T^2 = \mathbb{R}^2 / \mathbb{Z}^2$ for simplicity. Total automorphism $f = f_A$ induced by $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$$\text{Eigenvalues: } \lambda_u = \frac{1+\sqrt{5}}{2} > 1, \quad -1 < \lambda_s = \frac{1-\sqrt{5}}{2} < 0$$

$$\text{Eigenvektoren: } v^u = \left(1, \frac{-1+\sqrt{5}}{2} \right)^T, \quad v^s = \left(1, \frac{-1-\sqrt{5}}{2} \right)^T, \quad v^u \perp v^s,$$



~~→~~ $-1 < \lambda_s < 0$, \Rightarrow a directional reversal
and a contraction in V^S -component.

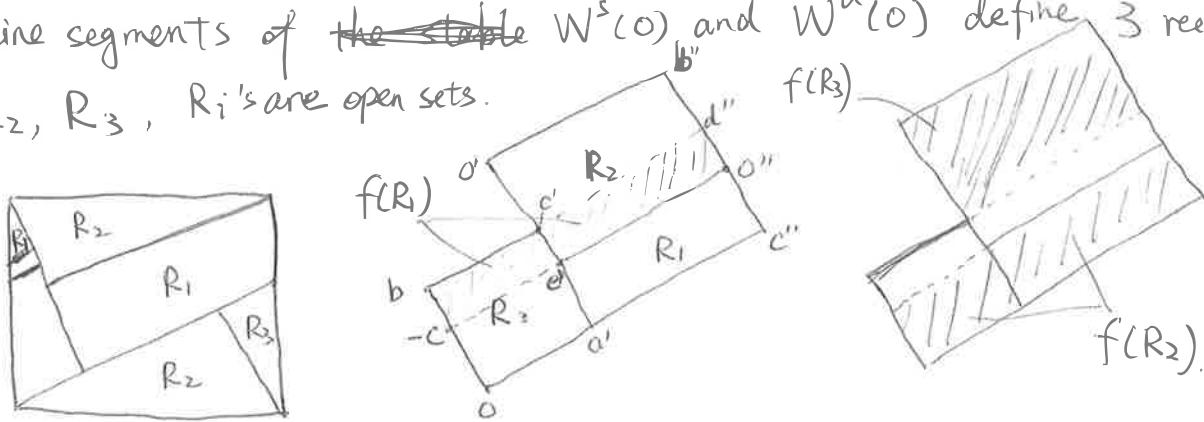
$\lambda_u > 1 \Rightarrow$ an expansion in V^u -component
 Take the part of stable manifold ~~from~~ from o downward to a , where it hit the unstable manifold.

Extend the stable manifold from o upward to b , where it hits the unstable manifold.

Basic geometry shows that $b = f(a)$. Let $c = f(b)$.

Extend the unstable manifold of o to the ~~upper left~~^{upward} and downward where it hits the stable manifolds at c'' and e .
 $c'' = c + (1, 1)$, ~~$e = -b + (-1, 0)$~~ , $e' = e + (1, 1)$.

The line segments of ~~the stable~~ $W^s(o)$ and $W^u(o)$ define 3 rectangles R_1, R_2, R_3 , R_i 's are open sets.



To determine $f(R_1)$, $f(R_2)$ and $f(R_3)$, we first determine ~~to~~ $f(a')$, $f(b')$, $f(c')$, $f(e')$. Note $a \equiv a' \equiv a'' \pmod{\mathbb{Z}^2}$, $b \equiv b' \equiv b'' \pmod{\mathbb{Z}^2}$, $c \equiv c' \equiv c'' \pmod{\mathbb{Z}^2}$, $e \equiv e' \equiv e'' \pmod{\mathbb{Z}^2}$.

Therefore $f(a') = b$, $f(b'') = c''$, $f(c'') = d''$, $f(e') = -c$, $f(c') = d''$
 $f(o) = o$, $f(o') = f(o'') = o = o' = o''$.

$$f(R_1) = b + c'' o'' d'', \quad f(R_2) = o + -c o'' c'', \quad f(R_3) = c' o' b'' d''.$$

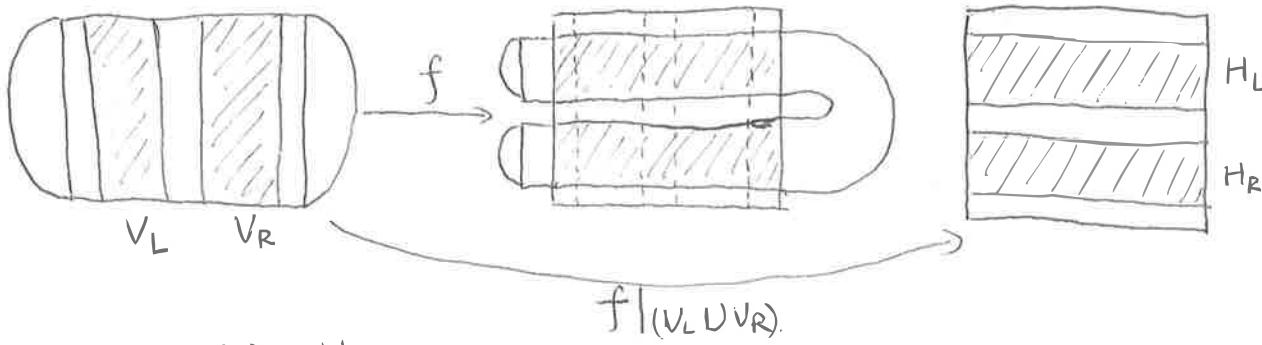
$f(R_1)$ meets R_2, R_3 but not R_1 .
 $f(R_2)$ meets R_1, R_3 but not R_2 .
 $f(R_3)$ meets R_2 only.

~~And the transition matrix is~~

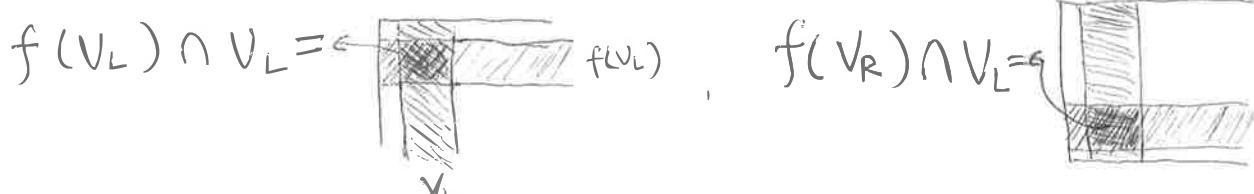
$$B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

By the last theorem, so $f \circ \pi = \pi \circ \sigma|_B$, $\pi: \Sigma_B \rightarrow \mathbb{T}^2$ is continuous, onto
and $= - -$

2. Horseshoe.



$$f(V_L) = H_L, \quad f(V_R) = H_R. \quad V_L, V_R \text{ are closed rectangles.}$$



All are good rectangle intersections. Also $V_L \cap V_R = \emptyset \Rightarrow \text{int } V_L \cap \text{int } V_R = \emptyset$.

~~Let $\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(V_L \cup V_R)$~~ . Then $(V_L \cap \Lambda) \cup (V_R \cap \Lambda) = \Lambda$.

$\{V_L \cap \Lambda, V_R \cap \Lambda\}$ is a Markov partition for ~~f~~ . $(\Lambda, f|_{(\Lambda \cap V_L \cup V_R)})$

The transition matrix is $A = \begin{bmatrix} L & R \\ R & L \end{bmatrix}$, since

3. Solenoid.

Consider solid torus $M = S^1 \times D$, D is the unit disk in \mathbb{R}^2 .

~~$f: M \rightarrow M$~~ , $f(\varphi, x, y) = 2\varphi, \frac{1}{10}x + \frac{1}{2}\cos\varphi, \frac{1}{10}y + \frac{1}{2}\sin\varphi$

f is well defined, i.e. ~~$f(x) \in M$~~ , $f(M) \subseteq M$.

Stable manifolds: cross sections $C_{\varphi_0} = \{(\varphi, x, y) \in X \mid \varphi = \varphi_0\}$, as φ_0 varies from ~~0~~ to 2π .

Unstable manifolds: A dense curve passing through every ~~$x \in M$~~ .

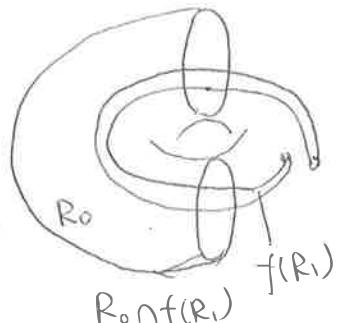
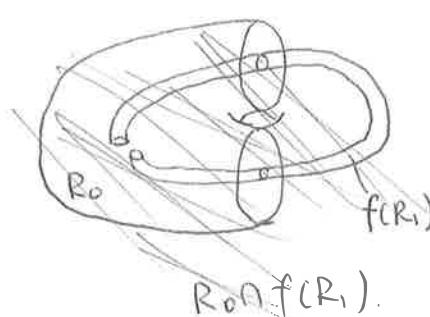
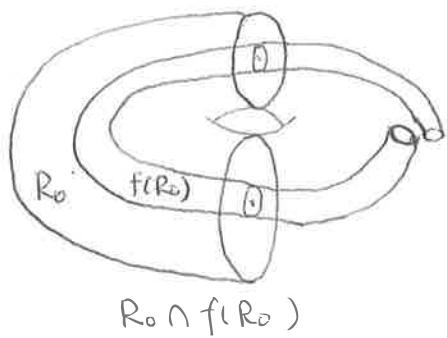
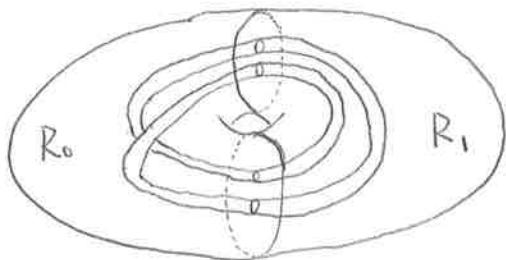
"dense" in $\Lambda = \bigcap_{n \in \mathbb{Z} \cup \{0\}} f^n(M)$.

Set $w = \varphi / 2\pi \bmod 1$.

$$\text{Let } R_0 = \{(w, x, y) \in \Lambda \mid 0 \leq w \leq \frac{1}{2}\}.$$

$$R_1 = \{(w, x, y) \in \Lambda \mid \frac{1}{2} \leq w \leq 1\}.$$

$$R_0 \cup R_1 = \Lambda, \quad \text{int } R_0 \cap \text{int } R_1 = \emptyset.$$



$\Rightarrow R_1 \cap f(R_1)$ and $R_0 \cap f(R_0)$ are similar.

All of the intersections are "good".

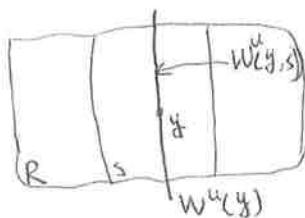
Then $\mathcal{R} = \{R_0, R_1\}$ is a Markov partition for Λ .

The transition matrix is $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Proof of Theorem 2:

$$\textcircled{1} \quad K_n(a) = \bigcap_{j=-n}^n f^{-j} R_a \neq \emptyset.$$

Def: Let R, S be two rectangles. S is called u -subrectangle of R if
 (a) $S \neq \emptyset$, $S \subset R$, S is proper, and (b) $W^u(y, S) = W^u(y, R)$ for $y \in S$.



Lemma: Suppose S is a u -subrectangle of R_i and $A_{ij} = 1$. Then $f(S) \cap R_j$ is a u -subrectangle of R_j .

$$\text{int}(R_i) \cap f^{-1} \text{int}(R_j) \neq \emptyset.$$

If $a_1 a_2 \dots a_n$ is a coding word with $A_{aj, aj+1} = 1$, then using the lemma inductively

one sees that

$$\bigcap_{j=1}^n f^{n-j} R_{aj} = R_n \cap f\left(\bigcap_{j=1}^{n-1} f^{n-1-j} R_{aj}\right), \quad n > 1.$$

is a n -subrectangle of R_n , since R_n is a ~~n -subrectangle non empty~~

Then ~~$K_n(a)$~~ . Since ~~R_n~~ is nonempty and proper,

$$\bigcap_{j=1}^n f^{-j} R_{aj} \neq \emptyset.$$

Then $K_n(a) = \bigcap_{j=-n}^n f^{-j} R_{aj} \neq \emptyset$. and ~~is the~~ $K_n(a) = \overline{\text{int}(K_n(a))}$

As $K_n(a) \supset K_{n+1}(a) \supset \dots$ we have

$$K(a) = \bigcap_{j=-\infty}^{\infty} f^{-j} R_{aj} = \bigcup_{n=1}^{\infty} K_n(a) \neq \emptyset.$$

(2) If $x, y \in K(a)$, then $f^j x, f^j y \in R_{aj}$ for all $j \in \mathbb{Z}$, ~~so $x=y$~~
since $d(f^j x, f^j y) < \delta <$ expansive constant for j , we have $x=y$.

(3) $K(\sigma a) = \bigcap_j f^{-j} R_{a_{j+1}} = f\left(\bigcap_j f^{-j} R_{aj}\right) = fK(a)$.

Then $\pi \circ \sigma = f \circ \pi$.

(4) π is continuous : If not, $\exists \{a_n\}, \{b_n\} \in \Sigma_A$, s.t. $a_{nj} = b_{nj}$ as $j \rightarrow \infty$,
but ~~$d(f^j(a_{nj}), f^j(b_{nj})) > \delta$~~ . $x = \pi a_n$, $y = \pi b_n$,
but $d(f^j(x), f^j(y)) > \delta$. This contradicts with the fact that
 $d(f^j(x), f^j(y)) < \varepsilon$ if $a_n = b_n$ for $|n| < N_0$, some $N_0 > 0$.

(5) ~~Since~~ Since R_i 's are proper, $\partial^s R \cup \partial^u R$ is nowhere dense,
 γ is residual. If $x \in \gamma$,

$f^j x \in \text{int}(R_{aj})$ & so $A_{aj} a_{j+1} = 1$. Thus $a = \{a_j\} \in \Sigma_A$, $x = \pi(a)$.

If $\pi(b) = x$, then $f^j x \in R_{bj}$ and $b_j = a_j$ because $f^j x \notin \partial^s R \cup \partial^u R$
and $\text{int}(R_{bj}) \cap \text{int}(R_{aj}) = \emptyset$. So π is 1-1.

As $\pi(\Sigma_A)$ is compact subset of Λ , containing a dense set γ ,
 $\pi(\Sigma_A) = \Lambda$.

