Lecture 35:
Recall that we are in the process of proving Bowen, Theorem 1.22.

Recall statements of Lemmas 1, 2 and 3 and defn. of Gibbs-Bowen measure.

Lemma 4: For $x, y \in \mathcal{M}_C^+$ with $x_i = y_i$ for $i = 0, \ldots, m - 1$,

$$|S_m \phi(x) - S_m \phi(y)| \leq a := \sum_{k=0}^{\infty} \text{var}_k(\phi).$$

Proof:

$$|S_m \phi(x) - S_m \phi(y)| \leq \sum_{i=0}^{m-1} |\phi(\sigma^i(x)) - \phi(\sigma^i(y))| \leq \sum_{i=0}^{m-1} \text{var}_{m-1-i} \leq a.$$

Lemma 5: $\mu := h\nu$ (from the Ruelle PF Theorem) is a Gibbs-Bowen measure for $\phi$.

Proof: Fix $x \in \mathcal{M}_C^+$. For any $z \in \mathcal{M}_C^+$, there is at most one $y \in \sigma^{-m}(z)$ s.t. $y \in E := E_{x_0 \ldots x_{m-1}}$. By Lemmas 0(a) and Lemma 3,

$$(\mathcal{L}^m(h\chi_E))(z) = \sum_{y \in \sigma^{-m}(z)} e^{S_m \phi(y)} h(y) \chi_E(y) \leq e^{S_m \phi(x)} e^a ||h||$$

Thus, with $\lambda$ as the eigenvalue from Ruelle PF,

$$\mu(E) = \nu(h\chi_E) = (1/\lambda^m) \mathcal{L}^* \nu(h\chi_E) = (1/\lambda^m) \int (\mathcal{L}^m(h \cdot \chi_E)) d\nu(z)$$

$$\leq (1/\lambda^m) e^{S_m \phi(x)} e^a ||h||$$

Letting $P := \log \lambda$ and $c_2 := e^a ||h||$, we obtain the upper bound in the defn of Bowen-Gibbs.
Let \( N \) be s.t. \( C^N > 0 \) (recall that \( C \) is primitive). For any \( z \in M_C^+ \), there is at least one \( y' \in \sigma^{-m-N}(z) \) s.t. \( y' \in E \). Then

\[
(L^{m+N}(h\chi_E))(z) \geq e^{S_m+n\phi(y')} h(y')
\]

\[
\geq e^{-N||\phi||-a(\inf h)} e^{S_m\phi(x)}
\]

So

\[
\mu(E) = \nu(h\chi_E) = (1/\lambda^{m+N}) \nu(L^{m+N}(h\chi_E)) \geq c_1 \lambda^{-m} e^{S_m\phi(x)}
\]

where \( c_1 := \lambda^{-N} e^{-N||\phi||-a(\inf h)} \); this gives the lower bound in the defn of Bowen-Gibbs. \( \square \)

Note: from Lemma 3 and proof of Lemma 5, we see that \( P(\phi) = P = \log \lambda \).

At this point, it can be shown that \( \mu := h\nu \) is the unique Gibbs-Bowen measure for \( \phi \) (Bowen, last part of Theorem 1.16). However, we will not need this fact and it will fall out later.

Lemma 6: Any Gibbs-Bowen measure for \( \phi \) is an equilibrium state for \( \phi \).

Proof: Let \( P = P(\phi) \). By the variational principle,

\[
h_\mu(\sigma|_{MC}) + \int \phi d\mu \leq P
\]

for all invariant measures \( \mu \) (in this case, this half of the variational principle is much easier than the general case).

Let \( \mu \) be a Gibbs-Bowen measure for \( \phi \).

For each nonempty initial cylinder set \( E := E_{a_0...a_{m-1}} \) in \( MC \) choose \( x = x_E \in E \) that achieves \( \min S_m\phi(x) \) on \( E \). Then

\[
-\mu(E) \log \mu(E) + \int_E S_m\phi \ d\mu \geq -\mu(E)(\log \mu(E) - S_m\phi(x_E))
\]
\[ \geq -\mu(E)(\log(c_2e^{-Pm+S_m\phi(x_E)}) - S_m\phi(x_E)) = \mu(E)(Pm - \log c_2) \]

So,

\[ H_\mu(\alpha^{[0,m-1]}) + \int S_m\phi d\mu \geq Pm - \log c_2 \]

So,

\[ h_\mu(\sigma|_{M_C}) + \int \phi d\mu = \lim_{m \to \infty} (1/m)(H_\mu(\alpha^{[0,m-1]}) + \int S_m\phi d\mu) \]

\[ \geq P \]

\[ \square \]

Note: Nishant will prove another version of Lemma 6 this afternoon.

Now we will prove uniqueness of the equilibrium state. We will need Lemmas 7,8,9.

Lemma 7: Given \( 0 < s \leq 1 \) and \( a_i \in \mathbb{R} \),

\[ -\sum_i p_i \log p_i + \sum_i p_i a_i \leq s(\log(\sum_i e^{a_i}) - \log s) \]

assuming \( p_i \geq 0 \), \( \sum_i p_i = s \).

Proof: After dividing by \( s \) and transposing \( -\log s \), the inequality above becomes

\[ -\sum_i (p_i/s) \log(p_i/s) + \sum_i (p_i/s)a_i \leq \log(\sum_i e^{a_i}) \]

subject to \( \sum_i p_i/s = 1 \). So this Lemma follows from our earlier lemma (page 4 of Lectures 30-32). \( \square \)

Lemma 8 (a special case of Bowen, Lemma 1.23): Let \( \mu_1 \perp \mu_2 \) be mutually singular invariant Borel probability measures on a two-sided shift space. Then there exists a sequence \( R_m \) of unions of elements of \( \alpha^{[0,m-1]} \) s.t. \( \mu_1(R_m) \to 1 \) and \( \mu_2(R_m) \to 0 \).
Proof: There is a Borel set $B$ s.t. $\mu_1(B) = 1$ and $\mu_2(B) = 0$.

Let $\epsilon > 0$ and let $K \subseteq B$ be a compact set s.t. $\mu_1(K) > 1 - \epsilon$. Let $R_m$ be the union of all elements of $\alpha[-m,m]$ which intersect $K$. Then, each $R_m$ contains $K$ and so $\mu_1(R_m) > 1 - \epsilon$.

Since the diameters of elements of $\alpha[-m,m]$ tend to zero and $K$ is compact, we have $R_m \downarrow K$.

So, $\mu_2(R_m) \to \mu_2(K) \leq \mu_2(B) = 0$. So, for some $m$, $\mu_2(R_m) < \epsilon$.

For a sequence $\epsilon_n \to 0$, choose corresponding $m_n$ s.t. $\mu_1(R_{m_n}) > 1 - \epsilon_n$ and $\mu_2(R_{m_n}) < \epsilon_n$.

By shifting the $R_{m_n}$ to the right, we get the desired sequence. □

Lemma 9: Let $T$ be a cts. map of a compact metric space and $\nu_1, \nu_2 \in \mathcal{M}(T)$. If $T$ is ergodic w.r.t. $\nu_1$ and $\nu_2 \ll \nu_1$, then $\nu_2 = \nu_1$.

Proof: By the Radon-Nikodym theorem, for some $f \in L^1(\nu_1)$, $f \geq 0$, we have $\nu_2 = f \nu_1$. Let $T^*$ be the induced action of $T$ on $\mathcal{M}(T)$. From the invariance of $\nu_1$ and $\nu_2$, one can show that $f$ is invariant, i.e., $f = f \circ T$ a.e. ($\nu_1$) – see Walters, pp. 152-153 for a version of this. Since $\mu_1$ is ergodic, $f$ is constant a.e. ($\nu_1$). Since $\nu_1$ and $\nu_2$ are both probability measures, the constant must be 1, and so $\nu_2 = \nu_1$. □

Nishant’s alternative proof of Lemma 9: First we claim that $\nu_2$ is also ergodic. To see this, first note that if $T^{-1}(A) = A$, then $\nu_1(A) = 0$ or 1. If $\nu_1(A) = 0$, then $\nu_2(A) = 0$; if $\nu_1(A) = 1$, then $\nu_1(A^c) = 0$ and so $\nu_2(A^c) = 0$ and so $\nu_2(A) = 1$.

Let $G$ be the set of $\nu_1$-generic points for a countable basis for the topology, i.e.,

$$G = \{x \in M : (1/n) \sum_{i=0}^{n-1} \chi_U(T^i x) \to \nu_1(U)\}$$
for all elements $U$ of the basis.

By the ergodic theorem, $\nu_1(G) = 1$ and so $\nu_2(G) = 1$ and so for a set of $\nu_2$-measure one, the frequency of visits to each basis element $U$ is $\nu_1(U)$; thus, by the ergodic theorem, $\nu_2(U) = \nu_1(U)$. □
Lecture 36:

Recall statement of Bowen’s Theorem 1.22, defn. of Gibbs-Bowen measure, fact that \( \mu := h\nu \) is Gibbs-Bowen and is an equilibrium state and the following lemmas:

Lemma 7: Given \( 0 < s \leq 1 \) and \( a_i \in \mathbb{R} \),

\[
- \sum_i p_i \log p_i + \sum_i p_i a_i \leq s \log(\sum_i e^{a_i}) - \log s
\]

assuming \( p_i \geq 0, \sum_i p_i = s \).

Lemma 8: Let \( \mu_1 \perp \mu_2 \) be mutually singular invariant Borel probability measures on a two-sided shift space. Then there exists a sequence \( R_m \) of unions of elements of \( \alpha^{[0,m-1]} \) s.t. \( \mu_1(R_m) \to 1 \) and \( \mu_2(R_m) \to 0 \).

Lemma 9: Let \( T \) be a cts. map of a compact metric space and \( \nu_1, \nu_2 \in \mathcal{M}(T) \). If \( T \) is ergodic w.r.t. \( \nu_1 \) and \( \nu_2 << \nu_1 \), then \( \nu_2 = \nu_1 \).

Proof of uniqueness of the equilibrium state: Let \( \lambda \) be an equilibrium state and \( \mu \), the Gibbs-Bowen measure given by Ruelle PF: \( \mu = h\nu \). Will show \( \lambda = \mu \).

DO NOT CONFUSE \( \lambda \) with the Ruelle PF eigenvalue.

All we need regarding \( \mu \) is that it is Gibbs-Bowen and ergodic.

Case 1: \( \lambda \perp \mu \)

Fix \( m \) and let \( R_m \) be as in Lemma 8, with \( \nu_1 = \lambda \) and \( \nu_2 = \mu \).

For each \( E \in \alpha^{[0,m-1]} \), choose \( x_E \in E \) that achieves \( \max_{x \in E} S_m \phi(x) \).

From now on, \( E \) implicitly means that \( E \in \alpha^{[0,m-1]} \).

We have

\[
mP = m h\lambda(\sigma|_{M_C}) + m \int \phi d\lambda \leq H\lambda(\alpha^{[0,m-1]}) + \int S_m \phi d\lambda
\]
\[ \leq \sum_{E \subseteq R_m} \lambda(E)(S_m \phi(x_E) - \log \lambda(E)) + \sum_{E \subseteq R_m^c} \lambda(E)(S_m \phi(x_E) - \log \lambda(E)) \]

which by Lemma 7 is

\[ \leq \lambda(R_m) \log \sum_{E \subseteq R_m} e^{S_m \phi(x_E)} + (\lambda(R_m^c)) \log \sum_{E \subseteq R_m^c} e^{S_m \phi(x_E)} + 2K^* \]

where \( K^* = \max_{s \in [0, 1]} -s \log s \). Thus

\[ -2K^* \leq \lambda(R_m) \log \sum_{E \subseteq R_m} e^{S_m \phi(x_E) - mP} + (\lambda(R_m^c)) \log \sum_{E \subseteq R_m^c} e^{S_m \phi(x_E) - mP} \]

and since \( \mu \) is a Gibbs-Bowen measure, this is

\[ \leq \lambda(R_m) \log c_1^{-1} \mu(R_m) + (\lambda(R_m^c)) \log c_1^{-1}(\mu(R_m^c)) \]

\[ \leq \log c_1^{-1} + \lambda(R_m) \log \mu(R_m) + (\lambda(R_m^c)) \log(\mu(R_m^c)) \]

But, as \( m \to \infty \), the 2nd term tends to \(-\infty\) and the third term tends to 0, a contradiction.

Perspective on proof in Case 1: Assume the special case that \( \phi = 0 \). Then \( P = P(\phi) = h_{\text{top}}(T) \).

Then the beginning of proof above shows: (letting \(|S|^*\) denote the number of cylinders sets of “span” \( m \) that comprise \( S \))

\[ mh_{\text{top}}(T) \leq \lambda(R_m) \log |R_m|^* + \lambda(R_m^c) \log |R_m^c|^* + 2K^* \quad (1) \]

Since \( \mu \) is a Gibbs-Bowen measure,

\[ |R_m|^* \approx e^{mh_{\text{top}}(T)} \mu(R_m), \quad |R_m^c|^* \approx e^{mh_{\text{top}}(T)} \mu(R_m^c), \]

Since \( \lambda(R_m) \to 1 \) and \( \mu(R_m) \to 0 \), we see that in \((??)\), the first term grows like \((1 + o(1))(mh_{\text{top}}(T) + \log o(1))\) and the second term grows like \(o(1))(mh_{\text{top}}(T)) + \log(1 + o(1))\).

Case 2: \( \lambda \) is not singular w.r.t. \( \mu \).
For an invariant measure $\rho$, write
\[ P_\rho(\phi) = h_\rho(T) + \int \phi d\rho \]
By the Lebesgue Radon Nikodym Decomposition, we can write $\lambda$ as the sum of two measures, one of which is mutually singular w.r.t. $\mu$ and the other absolutely continuous w.r.t. $\mu$. Since $\lambda$ is a probability measure, these two measures are finite measures, whose measures sum to 1. Thus, by normalizing we can write
\[ \lambda = \beta \lambda' + (1 - \beta) \mu' \]
where $\lambda' \perp \mu$, $\mu' << \mu$, $\beta \in [0, 1]$.

If $\beta = 0$, then $\lambda = \mu$ by Lemma 9. So we may assume $\beta \neq 0$.

We claim that $\lambda'$ and $\mu'$ are shift-invariant. To see this, first note that
\[ \lambda = \lambda \circ T^{-1} = \beta \lambda' \circ T^{-1} + (1 - \beta) \mu' \circ T^{-1} \]
By uniqueness of the Decomposition, it is enough to show that a) $\mu' \circ T^{-1} << \mu$ and b) $\lambda' \circ T^{-1} \perp \mu$.

a): If $\mu(E) = 0$, then $\mu(T^{-1}(E)) = 0$ and so $\mu' \circ T^{-1}(E) = 0$; so, $\mu' \circ T^{-1} << \mu$.

b): There is a Borel set $B$ s.t. $\lambda'(B) = 1$, $\mu(B) = 0$. Thus, $\lambda(B) = \beta \lambda'(B) + (1 - \beta) \mu'(B) = \beta$; this together with $\mu'(T^{-1}(B)) = 0$ gives:
\[ \beta = \lambda(B) = \lambda(T^{-1}(B)) = \beta \lambda'(T^{-1}(B)) + (1 - \beta) \mu'(T^{-1}(B)) = \beta \lambda'(T^{-1}(B)) \]
and so $\lambda'(T^{-1}(B)) = 1$; so, $\lambda' \circ T^{-1} \perp \mu$.

Since the entropy function is affine, we have
\[ P = P_\lambda(\phi) = \beta P_{\lambda'}(\phi) + (1 - \beta) P_{\mu'}(\phi) \]
Since $P_{\mu'}(\phi) \leq P$, $\beta \neq 0$ and by Case 1, $P_{\lambda'}(\phi) < P$, this is a contradiction. □