Lecture 35:

Recall that we are in the process of proving Bowen, Theorem 1.22.

Recall statements of Lemmas 1,2 and 3 and defn. of Gibbs-Bowen measure.

Lemma 4: For $x, y \in \mathcal{M}_C^+$ with $x_i = y_i$ for $i = 0, \ldots, m-1$,

$$|S_m\phi(x) - S_m\phi(y)| \leq a := \sum_{k=0}^{\infty} var_k(\phi).$$

Proof:

$$|S_m\phi(x) - S_m\phi(y)| \leq \sum_{i=0}^{m-1} |\phi(\sigma^i(x)) - \phi(\sigma^i(y))| \leq \sum_{i=0}^{m-1} var_{m-1-i} \leq a.$$

Lemma 5: $\mu := h\nu$ (from the Ruelle PF Theorem) is a Gibbs-Bowen measure for ϕ.

Proof: Fix $x \in M_C^+$.

For any $z \in M_C^+$, There is at most one $y \in \sigma^{-m}(z)$ s.t. $y \in E := E_{x_0 \ldots x_{m-1}}$. By Lemmas 0(a) and Lemma 3,

$$(\mathcal{L}^m(h\chi_E))(z) = \sum_{y \in \sigma^{-m}(z)} e^{S_m\phi(y)} h(y)\chi_E(y) \leq e^{S_m\phi(x)} e^a ||h||$$

Thus, with λ as the eigenvalue from Ruelle PF,

$$\mu(E) = \nu(h\chi_E) = (1/\lambda^m)\mathcal{L}^*\nu(h\chi_E) = (1/\lambda^m) \int (\mathcal{L}^m(h \cdot \chi_E)) d\nu(z)$$

$$\leq (1/\lambda^m) e^{S_m\phi(x)} e^a ||h||$$

Letting $P := \log \lambda$ and $c_2 := e^a ||h||$, we obtain the upper bound in the defn of Bowen-Gibbs.

Let N be s.t. $C^N > 0$ (recall that C is primitive). For any $z \in M_C^+$, there is at least one $y' \in \sigma^{-m-N}(z)$ s.t. $y' \in E$. Then

$$(\mathcal{L}^{m+N}(h\chi_E))(z) \geq e^{S_{m+N}\phi(y')}h(y')$$
$$\geq e^{-N||\phi||-a}(\inf h)e^{S_m\phi(x)}$$

So

$$\mu(E) = \nu(h\chi_E) = (1/\lambda^{m+N})\nu(\mathcal{L}^{m+N}(h\chi_E)) \geq c_1\lambda^{-m}e^{S_m\phi(x)}$$

where $c_1 := \lambda^{-N}e^{-N||\phi||-a}(\inf h)$; this gives the lower bound in the defn of Bowen-Gibbs. $\square$

Note: from Lemma 3 and proof of Lemma 5, we see that $P(\phi) = P = \log\lambda$.

At this point, it can be shown that $\mu := h\nu$ is the unique Gibbs-Bowen measure for ϕ (Bowen, last part of Theorem 1.16). However, we will not need this fact and it will fall out later.

Lemma 6: Any Gibbs-Bowen measure for ϕ is an equilibrium state for ϕ.

Proof: Let $P = P(\phi)$. By the variational principle,

$$h_\mu(\sigma|_{M_C}) + \int \phi d\mu \leq P$$

for all invariant measures μ (in this case, this half of the variational principle is much easier than the general case).

Let μ be a Gibbs-Bowen measure for ϕ.

For each nonempty initial cylinder set $E := E_{a_0\ldots a_{m-1}}$ in M_C choose $x = x_E \in E$ that achieves $\min S_m\phi(x)$ on E. Then

$$-\mu(E)\log\mu(E) + \int_E S_m\phi\, d\mu \geq -\mu(E)(\log\mu(E) - S_m\phi(x_E))$$

$$\geq -\mu(E)(\log(c_2 e^{-Pm+S_m\phi(x_E)}) - S_m\phi(x_E)) = \mu(E)(Pm - \log c_2)$$

So,

$$H_\mu(\alpha^{[0,m-1]}) + \int S_m\phi d\mu \geq Pm - \log c_2$$

So,

$$h_\mu(\sigma|_{M_C}) + \int \phi d\mu = \lim_{m\to\infty}(1/m)(H_\mu(\alpha^{[0,m-1]}) + \int S_m\phi d\mu)$$
$$\geq \lim_{m\to\infty}(1/m)(Pm - \log c_2) = P$$

□

Note: Nishant will prove another version of Lemma 6 this afternoon.

Now we will prove uniqueness of the equilibrium state. We will need Lemmas 7,8,9.

Lemma 7: Given $0 < s \leq 1$ and $a_i \in \mathbb{R}$,

$$-\sum_i p_i \log p_i + \sum_i p_i a_i \leq s(\log(\sum_i e^{a_i}) - \log s)$$

assuming $p_i \geq 0, \sum_i p_i = s$.

Proof: After dividing by s and transposing $-\log s$, the inequality above becomes

$$-\sum_i (p_i/s)\log(p_i/s) + \sum_i (p_i/s)a_i \leq \log(\sum_i e^{a_i}),$$

subject to $\sum_i p_i/s = 1$. So this Lemma follows from our earlier lemma (page 4 of Lectures 30-32). □

Lemma 8 (a special case of Bowen, Lemma 1.23): Let $\mu_1 \perp \mu_2$ be mutually singular invariant Borel probability measures on a two-sided shift space. Then there exists a sequence R_m of unions of elements of $\alpha^{[0,m-1]}$ s.t. $\mu_1(R_m) \to 1$ and $\mu_2(R_m) \to 0$.

Proof: There is a Borel set B s.t. $\mu_1(B) = 1$ and $\mu_2(B) = 0$.

Let $\epsilon > 0$ and let $K \subseteq B$ be a compact set s.t. $\mu_1(K) > 1 - \epsilon$. Let R_m be the union of all elements of $\alpha^{[-m,m]}$ which intersect K. Then, each R_m contains K and so $\mu_1(R_m) > 1 - \epsilon$.

Since the diameters of elements of $\alpha^{[-m,m]}$ tend to zero and K is compact, we have $R_m \downarrow K$.

So, $\mu_2(R_m) \to \mu_2(K) \leq \mu_2(B) = 0$. So, for some m, $\mu_2(R_m) < \epsilon$.

For a sequence $\epsilon_n \to 0$, choose corresponding m_n s.t. $\mu_1(R_{m_n}) > 1 - \epsilon_n$ and $\mu_2(R_{m_n}) < \epsilon_n$.

By shifting the R_{m_n} to the right, we get the desired sequence. $\square$

Lemma 9: Let T be a cts. map of a compact metric space and $\nu_1, \nu_2 \in \mathcal{M}(T)$. If T is ergodic w.r.t. ν_1 and $\nu_2 << \nu_1$, then $\nu_2 = \nu_1$.

Proof: By the Radon-Nikodym theorem, for some $f \in L^1(\nu_1)$, $f \geq 0$, we have $\nu_2 = f\nu_1$. Let T^* be the induced action of T on $\mathcal{M}(T)$. From the invariance of ν_1 and ν_2, one can show that f is invariant, i.e., $f = f \circ T$ a.e. (ν_1) – see Walters, pp. 152-153 for a version of this. Since μ_1 is ergodic, f is constant a.e. (ν_1). Since ν_1 and ν_2 are both probability measures, the constant must be 1, and so $\nu_2 = \nu_1$. $\square$

Nishant's alternative proof of Lemma 9: First we claim that ν_2 is also ergodic. To see this, first note that if $T^{-1}(A) = A$, then $\nu_1(A) = 0$ or 1. If $\nu_1(A) = 0$, then $\nu_2(A) = 0$; if $\nu_1(A) = 1$, then $\nu_1(A^c) = 0$ and so $\nu_2(A^c) = 0$ and so $\nu_2(A) = 1$.

Let G be the set of ν_1-generic points for a countable basis for the topology, i.e.,

$$G = \{x \in M : (1/n) \sum_{i=0}^{n-1} \chi_U(T^i x) \to \nu_1(U)\}$$

for all elements U of the basis.

By the ergodic theorem, $\nu_1(G) = 1$ and so $\nu_2(G) = 1$ and so for a set of ν_2-measure one, the frequency of visits to each basis element U is $\nu_1(U)$; thus, by the ergodic theorem, $\nu_2(U) = \nu_1(U)$. $\square$

Lecture 36:

Recall statement of Bowen's Theorem 1.22, defn. of Gibbs-Bowen measure, fact that $\mu := h\nu$ is Gibbs-Bowen and is an equilibrium state and the following lemmas:

Lemma 7: Given $0 < s \leq 1$ and $a_i \in \mathbb{R}$,

$$-\sum_i p_i \log p_i + \sum_i p_i a_i \leq s(\log(\sum_i e^{a_i}) - \log s)$$

assuming $p_i \geq 0, \sum_i p_i = s$.

Lemma 8: Let $\mu_1 \perp \mu_2$ be mutually singular invariant Borel probability measures on a two-sided shift space. Then there exists a sequence R_m of unions of elements of $\alpha^{[0,m-1]}$ s.t. $\mu_1(R_m) \to 1$ and $\mu_2(R_m) \to 0$.

Lemma 9: Let T be a cts. map of a compact metric space and $\nu_1, \nu_2 \in \mathcal{M}(T)$. If T is ergodic w.r.t. ν_1 and $\nu_2 << \nu_1$, then $\nu_2 = \nu_1$.

Proof of uniqueness of the equilibrium state: Let λ be an equilibrium state and μ, the Gibbs-Bowen measure given by Ruelle PF: $\mu = h\nu$. Will show $\lambda = \mu$.

DO NOT CONFUSE λ with the Ruelle PF eigenvalue.

All we need regarding μ is that it is Gibbs-Bowen and ergodic.

Case 1: $\lambda \perp \mu$

Fix m and let R_m be as in Lemma 8, with $\nu_1 = \lambda$ and $\nu_2 = \mu$.

For each $E \in \alpha^{[0,m-1]}$, choose $x_E \in E$ that achieves $\max_{x \in E} S_m\phi(x)$.

From now on, E implicitly means that $E \in \alpha^{[0,m-1]}$.

We have

$$mP = mh_\lambda(\sigma|_{M_C}) + m\int \phi d\lambda \leq H_\lambda(\alpha^{[0,m-1]}) + \int S_m\phi d\lambda$$

$$\leq \sum_{E\subseteq R_m} \lambda(E)(S_m\phi(x_E)-\log\lambda(E))+\sum_{E\subseteq R_m^c} \lambda(E)(S_m\phi(x_E)-\log\lambda(E))$$

which by Lemma 7 is

$$\leq \lambda(R_m)\log\sum_{E\subseteq R_m} e^{S_m\phi(x_E)} + (\lambda(R_m^c))\log\sum_{E\subseteq R_m^c} e^{S_m\phi(x_E)} + 2K^*$$

where $K^* = \max_{s\in[0,1]} -s\log s$. Thus

$$-2K^* \leq \lambda(R_m)\log\sum_{E\subseteq R_m} e^{S_m\phi(x_E)-mP}+(\lambda(R_m^c))\log\sum_{E\subseteq R_m^c} e^{S_m\phi(x_E)-mP}$$

and since μ is a Gibbs-Bowen measure, this is

$$\leq \lambda(R_m)\log c_1^{-1}\mu(R_m) + (\lambda(R_m^c))\log c_1^{-1}(\mu(R_m^c))$$

$$\leq \log c_1^{-1} + \lambda(R_m)\log\mu(R_m) + (\lambda(R_m^c))\log(\mu(R_m^c))$$

But, as $m\to\infty$, the 2nd term tends to $-\infty$ and the third term tends to 0, a contradiction.

Perspective on proof in Case 1: Assume the special case that $\phi = 0$. Then $P = P(\phi) = h_{top}(T)$.

Then the beginning of proof above shows: (letting $|S|^*$ denote the number of cylinders sets of "span" m that comprise S)

$$mh_{top}(T) \leq \lambda(R_m)\log|R_m|^* + \lambda(R_m^c)\log|R_m^c|^* + 2K^* \qquad (1)$$

Since μ is a Gibbs-Bowen measure,

$$|R_m|^* \approx e^{mh_{top}(T)}\mu(R_m), \quad |R_m^c|^* \approx e^{mh_{top}(T)}\mu(R_m^c),$$

Since $\lambda(R_m)\to 1$ and $\mu(R_m)\to 0$, we see that in (**??**), the first term grows like $(1+o(1))(mh_{top}(T)+\log o(1))$ and the second term grows like $(o(1))((mh_{top}(T))+\log(1+o(1))$.

Case 2: λ is not singular w.r.t. μ.

For an invariant measure ρ, write

$$P_\rho(\phi) = h_\rho(T) + \int \phi d\rho$$

By the Lebesgue Radon Nikodym Decomposition, we can write λ as the sum of two measures, one of which is mutually singular w.r.t. μ and the other absolutely continuous w.r.t. μ. Since λ is a probability measure, these two measures are finite measures, whose measures sum to 1. Thus, by normalizing we can write

$$\lambda = \beta\lambda' + (1-\beta)\mu'$$

where $\lambda' \perp \mu,\ \mu' << \mu,\ \beta \in [0,1]$.

If $\beta = 0$, then $\lambda = \mu$ by Lemma 9. So we may assume $\beta \neq 0$.

We claim that λ' and μ' are shift-invariant. To see this, first note that

$$\lambda = \lambda \circ T^{-1} = \beta\lambda' \circ T^{-1} + (1-\beta)\mu' \circ T^{-1}$$

By uniqueness of the Decomposition, it is enough to show that a) $\mu' \circ T^{-1} << \mu$ and b) $\lambda' \circ T^{-1} \perp \mu$.

a): If $\mu(E) = 0$, then $\mu(T^{-1}(E) = 0$ and so $\mu' \circ T^{-1}(E) = 0$; so, $\mu' \circ T^{-1} << \mu$.

b): There is a Borel set B s.t. $\lambda'(B) = 1, \mu(B) = 0$. Thus, $\lambda(B) = \beta\lambda'(B) + (1-\beta)\mu'(B) = \beta$; this together with $\mu'(T^{-1}(B) = 0$ gives:

$$\beta = \lambda(B) = \lambda(T^{-1}(B)) = \beta\lambda'(T^{-1}(B)) + (1-\beta)(\mu'(T^{-1}(B)) = \beta\lambda'(T^{-1}(B))$$

and so $\lambda'(T^{-1}(B)) = 1$; so, $\lambda' \circ T^{-1} \perp \mu$.

Since the entropy function is affine, we have

$$P = P_\lambda(\phi) = \beta P_{\lambda'}(\phi) + (1-\beta)P_{\mu'}(\phi)$$

Since $P_{\mu'}(\phi) \leq P$, $\beta \neq 0$ and by Case 1, $P_{\lambda'}(\phi) < P$, this is a contradiction. $\square$