Two types of "chaotic" behaviour

Ex.

Regional: Positive Entropy.

Everywhere chaos:

- Mixing
- K-system (positive entropy for any partition)

Recall Let $\Sigma_+ :=$ sets of positive measure

Mixing:

$$\forall A, B \in \Sigma_+$$

$$\lim_{n\to\infty} \mu(A \cap T^{-n}B) = \mu(A)\mu(B).$$

Def: 3-mixing $\forall A, B, C \in \Sigma_+$

$$\lim_{n_1, n_2 \to \infty} \mu(A \cap T^{-n_1} B \cap T^{-n_2} C) = \mu(A)\mu(B)\mu(C)$$

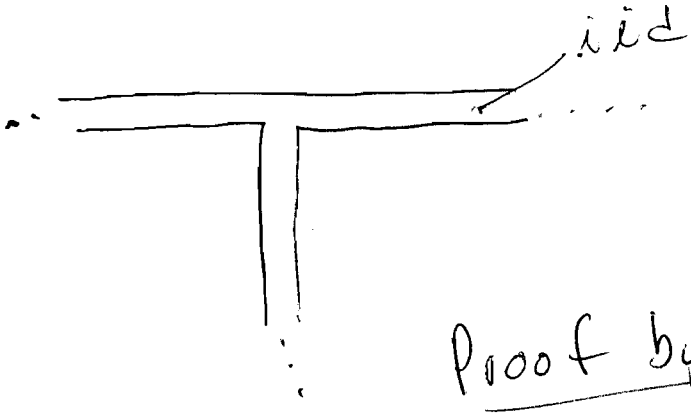
Question: Does mixing imply 3-mixing?

Rokhlin 1949

1D-unknown 2D-counterexample.

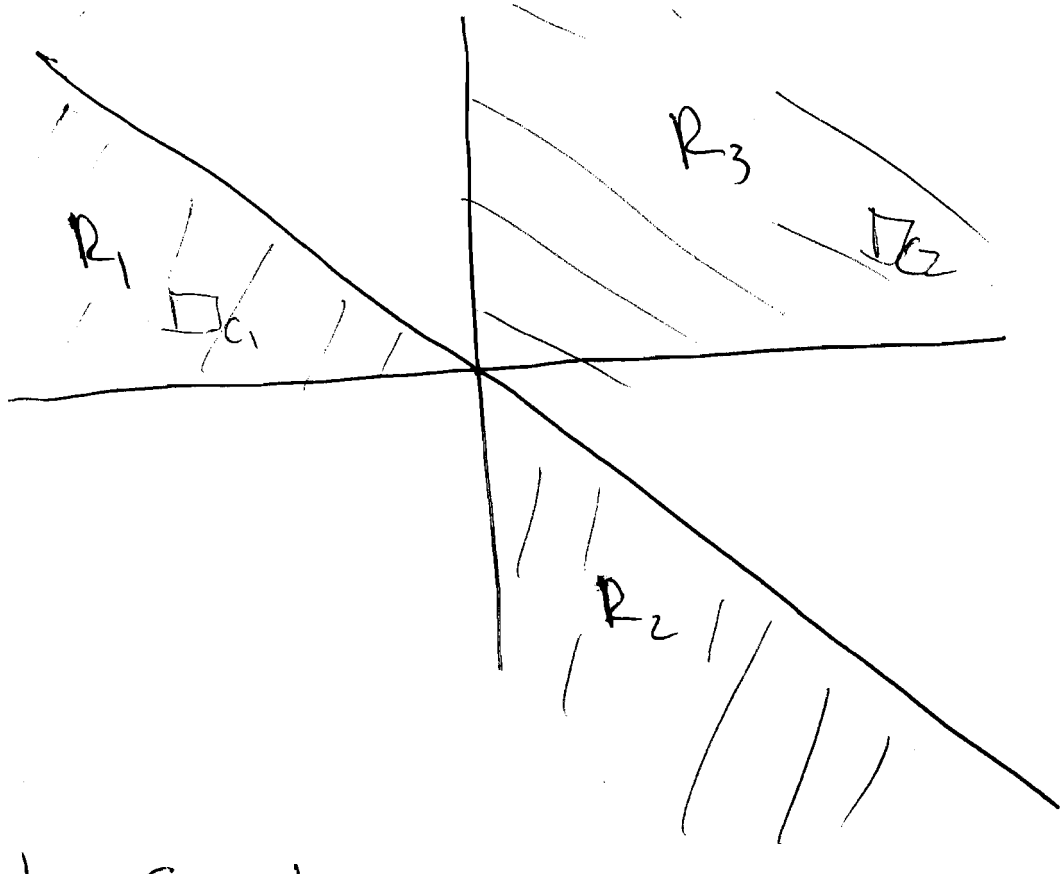
• Ledrappier 3 dot example. $\begin{array}{|c|c|}\hline a & \\ \hline b & c \\ \hline \end{array}$ $a+b+c = 0 \bmod 2$

Measure μ. defined by iid. on the T



Proof by picture

• Three independent regions



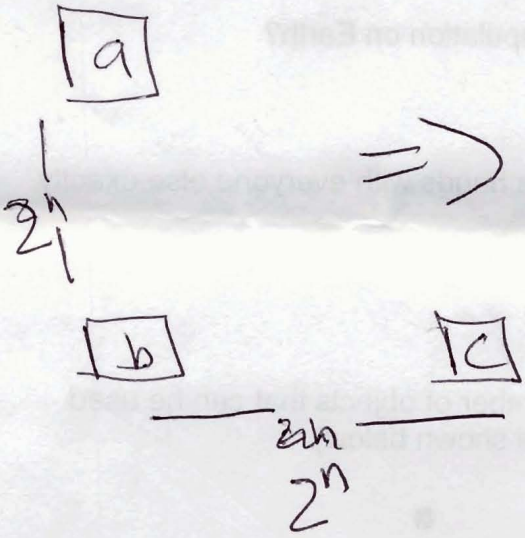
Let C, C' be cylinder sets
if they are sufficiently far they can be

shifted to be in two different regions off (which are independent).

[stronger than mixing!].

On the other hand.

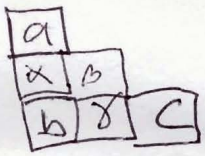
Lemma



$\Rightarrow a+b+c=0 \bmod 2.$

$n=0$ hypothesis.

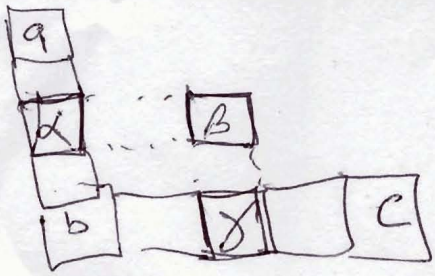
$n=1$



$$\left.\begin{array}{l} a+\alpha+\beta=0 \\ \alpha+b+\gamma=0 \\ \beta+\gamma+c=0 \end{array}\right\} \text{2 } \alpha\text{'s}, \beta\text{'s}, \gamma\text{'s}$$

$$a+b+c=0.$$

$n>2$



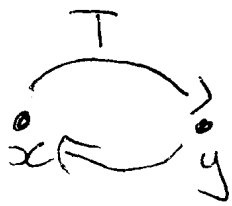
continue by induction ...

Product system.

$$T \times T : X^2 \to X^2$$

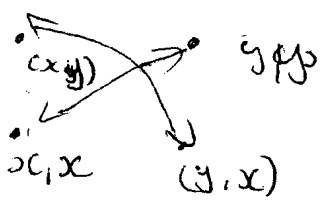
$$T \times T(x, y) = (Tx, Ty).$$

Example.



$\mu(x) = \mu(y) = 1/2$

Product



$T \times T(x, x) = (y, y)$

<u>Not Ergodic</u>

$T \times T$ ergodic implies "mixing" of some form.

- Proposition T is mixing iff $T \times T$ is mixing.

<u>Proof</u>: Let $A, B, C, D \in \Sigma_+$

- $\lim \mu(T^{-n} A \cap B) = \mu(A)\mu(B)$
- $\lim \mu(T^{-n} C \cap D) = \mu(C)\mu(D)$

$\Rightarrow$

$$\lim \mu\times\mu\,(T\times T^{-n}(A\times C)\cap(B\times D))$$

$$=\lim \mu(T^{-n}A\cap B)\,\mu(T^{-n}C\cap B)$$

$$=\mu(A)\mu(B)\mu(C)\mu(D)$$

$$=\mu\times\mu(A\times C)\cdot\mu\times\mu(B\times D).$$

Since rectangles form a ~~sem~~-generating semi-algebra we conclude $T\times T$ is mixing.

Corollary if T is mixing $T\times T\times T\ldots T$ is mixing.

Recall: Density on $\mathbb{Z}$.

We will focus on invertible MPT's i.e. $\mathbb{Z}$-systems.

Actually the result holds for any MPT not just mixing.

~~Thm~~ Erdös conjecture. (1936)

Let $J \subset \mathbb{Z}$ be a set with positive upper density. Then J contains arbitrarily large arithmetic progressions, $(n, n+r, \ldots (k-1)r)$

- Proved by Szemeredi 1975.
- 1976 Ergodic proof by Furstenberg.

Th. Multiple recurrence implies Szemeredi's theorem

Using the previous proposition and Van der Corput's Lemma.

It can be shown that

If (X, μ, T) is mixing then

There for any $A_1, A_2 \ldots A_k$

There exists $J \subset \mathbb{Z}$ with zero density such that

$$\lim_{\substack{n \to \infty \\ n \notin J}} \mu(A_1 \cap T^n A_2 \cap T^{2n} A_3 \ldots \cap T^{n(k-1)} A_k) = \mu(A_1)\mu(A_2)\ldots\mu(A_k).$$

Which is some form of higher order mixing.

<u>Corollary</u> (Recurrence Theorem for mixing)

Let $k \in \mathbb{Z}_+$ and (X, μ, T) mixing. and $E \in \Sigma_+$

There exists $r > 0$ such that

$E \cap T^{-r} E \ldots \cap T^{(k-1)r} E$ is non-empty.

Proof

Let $J \subset \mathbb{Z}$ with positive upper density. Let $x \in \{0,1\}^{\mathbb{Z}}$ be such that $x_i = 1$ iff $i \in J$.

$X = \overline{\{\sigma^n x : n \in \mathbb{Z}\}}$ be a shifspace.

$E = \{x_0 = 1\}$ cylinder set $\subset X$

If there exists an invariant measure μ such that $\mu(E) > 0$ then by multiple recurrence for every $k \exists n$ s.t.

$F = E \cap \sigma^n E \ldots \cap \sigma^{(k-1)n} E \neq \emptyset$.

- There exists m such that

$\sigma^{-m} x \in F$, Hence J contains a k-arithmetic progression.