Two types of "chaotic" behaviour

*Ex.* Regional: Positive Entropy.

Everywhere chaos:
- Mixing
- K-system (positive entropy for any partition)

Recall Let \( \Sigma_+ \) = sets of positive measure

**Mixing:**

\[ \forall A, B \in \Sigma_+ \]

\[ \lim_{n \to \infty} \mu(A \cap T^{-n} B) = \mu(A) \mu(B) \]

**Def:** 3-mixing \( \forall A, B, C \in \Sigma_+ \)

\[ \lim_{n_1, n_2 \to \infty} \mu(A \cap T^{-n_1} B \cap T^{-n_2} C) = \mu(A) \mu(B) \mu(C) \]

**Question:** Does mixing imply 3-mixing?

Rokhlin 1949

1D-unknown 2D-counterexample.
Ledrappier's dot example:

\[
\begin{array}{c|c|c}
\text{a} & \text{b} & \text{c} \\
\hline
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
\end{array}
\] \text{ mod } 2

Measure \( \mu \) defined by \( \text{i.i.d.} \) on the \( T \)

- Three independent regions \( S \)

Proof by picture

Let \( C, C' \) be cylinder sets if they are sufficiently far they can be
shifted to be in two different regions of (which are independent!)

[stronger than mixing!]

On the other hand.

Lemma

\[ a \quad \text{mod} \quad 2 \]

\[ b \quad c \]

\[ 2^n \]

\[ a + b + c = 0 \quad \text{mod} \quad 2 \]

\[ n = 0 \quad \text{hypothesis} \]

\[ n = 1 \]

\[ a \]

\[ x, y, z \]

\[ a + x + y = 0 \]

\[ a + b + x = 0 \]

\[ \beta + y + c = 0 \]

\[ a + b + c = 0 \]

\[ n = 2 \]

\[ a \quad b \quad c \]

\[ \text{Continue by induction} \ldots \]
Product system.

\( T \times T : X^2 \to X^2 \)

\( T \times T (x, y) = (Tx, Ty) \).

Example.

\[ T \]

\[ \begin{array}{c}
  x \rightarrow y \\
  x = y
\end{array} \]

\( M(x) = M(y) = 1/2 \)

Product

\[ \begin{array}{c}
  (x, y) \rightarrow \text{gap} \\
  (x, x) \rightarrow (y, x)
\end{array} \]

\( T \times T (x, x) = (y, y) \)

Not Ergodic

\( T \times T \) ergodic implies "mixing" of some form.

- Proposition: \( T \) is mixing iff \( T \times T \) is mixing.

Proof: Let \( A, B, C, D \in \Sigma^+ \).

- \( \lim M(T^{-n} A \land B) = M(A) M(B) \)
- \( \lim M(T^{-n} C \land D) = M(C) M(D) \)
\[
\lim_{n \to \infty} M X M (T^n A \cap B) = M(\mathcal{A}) M(\mathcal{B}) M(\mathcal{C}) M(\mathcal{D}) = M X M (A X C) \cdot M X M (B X D).
\]

Since rectangles form a semi-generating semi-algebra, we conclude that \(T X T\) is mixing.

**Corollary:** If \(T\) is mixing, then \(T X T X T \ldots T\) is mixing.

**Recall:** Density on \(Z\).

We will focus on invertible MFT's, i.e., \(Z\)-systems.
Actually the result holds for any MPT not just mixing.

\[ \text{Thm} \]
Erdős' conjecture. (1936)

Let \( J \subseteq \mathbb{Z} \) be a set with positive upper density. Then \( J \) contains arbitrarily large arithmetic progressions, \( (n, n + r, \ldots, (k-1)r) \).

* Proved by Szemerédi 1975.

* 1976 Ergodic proof by Furstenberg.

Th. Multiple recurrence implies Szemerédi's theorem.
Using the previous proposition and Van der Corput's Lemma.

It can be shown that

If \((X,\mathcal{M},\mu,\tau)\) is mixing then

for any \(A_1, A_2, \ldots, A_k\)

there exists \(J \in \mathbb{Z}^\omega\) with zero density such that

\[
\lim_{n \to \infty} \mu(\bigcap A_1 \bigcap T^{n} A_2 \bigcap \cdots \bigcap T^{n(k-1)} A_k) \\

n \notin J \\
= \mu(A_1) \mu(A_2) \cdots \mu(A_k).
\]

Which is some form of higher order mixing.

Corollary (Recurrent Theorem for mixing)

Let \(k \in \mathbb{Z}^\omega\) and \((X,\mathcal{M},\mu,\tau)\) mixing, and \(E \in \mathcal{E} \subset \mathbb{E}^+\)

there exists \(r > 0\) such that

\[
E \cap T^{-r} E \cap \cdots \cap T^{-(k-1)r} E \text{ is non-empty.}
\]
Proof

Let $J \subset \mathbb{Z}$ with positive upper density. Let $x \in \sigma^J \mathbb{Z}$ be such that $x_i = 1$ iff $i \in J$.

$$X = \sigma^J \mathbb{Z}$$ be a shift space.

$$E = \{ x_0 = 1 \}$$ cylinder set. $C_X$

If there exists an invariant measure $\mu$ such that $\mu(E) > 0$ then by multiple recurrence for every $k \in \mathbb{N}

$$S^k F = E \cap \sigma^k E \ldots \cap \sigma^{k-1} E \neq \emptyset$$

There exists $m$ such that

$$\sigma^m x \in F \text{, hence } J$$ contains a $k$-arithmetic progression.