Running list of HW2 Problems:

1. Show that the following are equivalent for a measure preserving $\mathbb{Z}_+^d$ action $\bar{i} \mapsto T^{\bar{i}}$ on a probability space $(M, \mathcal{A}, \mu)$:

   (a) whenever $A \in \mathcal{A}$ and $\mu(A) > 0$, then $\mu((\cup_{\bar{i} \in \mathbb{Z}_+^d} T^{-\bar{i}}(A)) = 1$

   (b) if $A, B \in \mathcal{A}$ and $\mu(A), \mu(B) > 0$, there exists $\overline{n} \in \mathbb{Z}_+^d \setminus \{0\}$ s.t. $\mu(T^{-\overline{n}}(A) \cap B) > 0$.

   (c) $\bar{i} \mapsto T^{\bar{i}}$ is ergodic, i.e. if $A \in \mathcal{A}$ and $T^{-\bar{i}}(A) = A$ for all $\bar{i} \in \mathbb{Z}_+^d$, then $\mu(A) = 0$ or 1.

   (d) if $A \in \mathcal{A}$ and $\mu(T^{-\bar{i}}(A) \Delta A) = 0$ for all $\bar{i} \in \mathbb{Z}_+^d$, then $\mu(A) = 0$ or 1.

   (e) If $f \in L^p$ and $f \circ T^{\bar{i}} = f$ a.e. for all $\bar{i} \in \mathbb{Z}_+^d$, then f is constant a.e. (here, $0 \leq p \leq \infty$)

   (f) if $A, B \in \mathcal{A}$, then

   $$\lim_{n \to \infty} (1/n^d) \sum_{\bar{i} \in \mathbb{Z}_+^d, |\bar{i}| \leq n} \mu(T^{-\bar{i}}(A) \cap B) = \mu(A)\mu(B)$$

   (here, $|\bar{i}|$ is the L^∞ norm)

2. (a) An $\mathbb{Z}_+^d$ action is *totally ergodic* if $T^{\bar{i}}$ is ergodic for all nonzero $\bar{i} \in \mathbb{Z}_+^d$. Show that the action is totally ergodic $\Rightarrow$ Some element is ergodic $\Rightarrow$ the action is ergodic

   (b) A $\mathbb{Z}_+^d$ action is *totally mixing* if $T^{\bar{i}}$ is mixing for all nonzero $\bar{i} \in \mathbb{Z}_+^d$. Show that the action is mixing $\Rightarrow$ the action is totally mixing $\Rightarrow$ Some element is mixing.

3. (a) Show that a $\mathbb{Z}^d$ action is measure-preserving as a $\mathbb{Z}^d$ action iff it is measure-preserving as a $\mathbb{Z}_+^d$ action.

   (b) Show that a measure-preserving $\mathbb{Z}^d$ action is ergodic as a $\mathbb{Z}^d$ action iff it is ergodic as a $\mathbb{Z}_+^d$ action.

4. Given that $\gamma \preceq \alpha$ and $H(\beta|\alpha) = H(\beta|\gamma)$, what can you conclude about α, β and γ?

5. Let α, β, γ and δ be finite measurable partitions of a probability space. Prove:

   (a) $H(\alpha \vee \beta|\gamma) = H(\alpha|\gamma) + H(\beta|\alpha, \gamma)$

   (b) $H(\beta|\gamma, \alpha) \leq H(\beta|\gamma)$ with equality iff $\alpha \perp_\gamma \beta$, i.e., for each element C of γ, the restrictions of α to C and β to C are independent.

   (c) If $\alpha \perp_\gamma (\beta, \delta)$, then $H(\beta|\delta, \gamma, \alpha) = H(\beta|\delta, \gamma)$.

   (d) If $\gamma \preceq \alpha$, then $H(\gamma|\beta) \leq H(\alpha|\beta)$

   (e) If T is an MPT, then for all i, j, k,
   $H(T^{-i}\alpha \vee \ldots \vee T^{-j}\alpha) = H(T^{-i-k}\alpha \vee \ldots \vee T^{-j-k}\alpha)$

6. Prove the entropy inequalities (conditioning on sub-σ-algebras): see page 5 of Lectures 18-20 on course website).

7. True or False:
$$I_{\alpha|\mathcal{B}} = E(I_\alpha|\mathcal{B})$$

8. Prove the two-sided generator theorem by modifying the proof of the one-sided generator theorem.

9. For an MPT T and positive integer n show that $h(T^n) = nh(T)$.

10. Let $\mathcal{T}$ be a $\mathbb{Z}^d$ action and let $\mathcal{T}^{-1}$ be the $\mathbb{Z}^d$ action defined by $\bar{i} \mapsto T^{-\bar{i}}$. Show that $h(\mathcal{T}^{-1}) = h(\mathcal{T})$

11. Show that if an MP $\mathbb{Z}^d$ action $\mathcal{T}$ has a one-sided generator, then $h(\mathcal{T}) = 0$.

12. Compute the entropy of a rotation of the circle.

13. For each of the following, give one-sided generators and two-sided generators (whenever they make sense and exist):

    (a) The doubling map

    (b) The Baker's transformation

14. Let $\mathcal{T}$ be an MP $\mathbb{Z}^d_+$ action with $d \geq 2$. Suppose that for some $\bar{i} \in \mathbb{Z}^d_+ \setminus \{\bar{0}\}$, $h(T^{\bar{i}}) < \infty$. Show that $h(\mathcal{T}) = 0$.