Running list of HW2 Problems:

1. Show that the following are equivalent for a measure preserving $\mathbb{Z}^d$ action $\tilde{t} \mapsto T\tilde{t}$ on a probability space $(M, \mathcal{A}, \mu)$:

   (a) whenever $A \in \mathcal{A}$ and $\mu(A) > 0$, then $\mu((\bigcup_{t \in \mathbb{Z}^d_+} T^{-t}A)) = 1$

   (b) if $A, B \in \mathcal{A}$ and $\mu(A), \mu(B) > 0$, there exists $\pi \in \mathbb{Z}^d_+ \setminus \{0\}$ s.t. $\mu(T^{-\pi}(A) \cap B) > 0$.

   (c) $\tilde{t} \mapsto T\tilde{t}$ is ergodic, i.e. if $A \in \mathcal{A}$ and $T^{-\tilde{t}}(A) = A$ for all $\tilde{t} \in \mathbb{Z}^d_+$, then $\mu(A) = 0$ or 1.

   (d) if $A \in \mathcal{A}$ and $\mu(T^{-\tilde{t}}(A) \Delta A) = 0$ for all $\tilde{t} \in \mathbb{Z}^d_+$, then $\mu(A) = 0$ or 1.

   (e) if $f \in L^p$ and $f \circ T\tilde{t} = f$ a.e. for all $\tilde{t} \in \mathbb{Z}^d_+$, then $f$ is constant a.e. (here, $0 \leq p \leq \infty$)

   (f) if $A, B \in \mathcal{A}$, then

   $$\lim_{n \to \infty} \frac{1}{n^d} \sum_{\tilde{t} \in \mathbb{Z}^d_+, |\tilde{t}| \leq n} \mu(T^{-\tilde{t}}(A) \cap B) = \mu(A)\mu(B)$$

   (here, $|\tilde{t}|$ is the $L^\infty$ norm)

2. (a) An $\mathbb{Z}^d_+$ action is **totally ergodic** if $T\tilde{t}$ is ergodic for all nonzero $\tilde{t} \in \mathbb{Z}^d_+$. Show that the action is totally ergodic $\Rightarrow$ Some element is ergodic $\Rightarrow$ the action is ergodic

   (b) A $\mathbb{Z}^d_+$ action is **totally mixing** if $T\tilde{t}$ is mixing for all nonzero $\tilde{t} \in \mathbb{Z}^d_+$. Show that the action is mixing $\Rightarrow$ the action is totally mixing $\Rightarrow$ Some element is mixing.

3. (a) Show that a $\mathbb{Z}^d$ action is measure-preserving as a $\mathbb{Z}^d$ action iff it is measure-preserving as a $\mathbb{Z}^d_+$ action.

   (b) Show that a measure-preserving $\mathbb{Z}^d$ action is ergodic as a $\mathbb{Z}^d$ action iff it is ergodic as a $\mathbb{Z}^d_+$ action.

4. Given that $\gamma \preceq \alpha$ and $H(\beta|\alpha) = H(\beta|\gamma)$, what can you conclude about $\alpha, \beta$ and $\gamma$?

5. Let $\alpha, \beta, \gamma$ and $\delta$ be finite measurable partitions of a probability space. Prove:

   (a) $H(\alpha \vee \beta|\gamma) = H(\alpha|\gamma) + H(\beta|\alpha, \gamma)$

   (b) $H(\beta|\gamma, \alpha) \leq H(\beta|\gamma)$ with equality iff $\alpha \perp_\gamma \beta$, i.e., for each element $C$ of $\gamma$, the restrictions of $\alpha$ to $C$ and $\beta$ to $C$ are independent.

   (c) If $\alpha \perp_\gamma (\beta, \delta)$, then $H(\beta|\delta, \gamma, \alpha) = H(\beta|\delta, \gamma)$.

   (d) If $\gamma \preceq \alpha$, then $H(\gamma|\beta) \leq H(\alpha|\beta)$

   (e) If $T$ is an MPT, then for all $i, j, k$,

   $$H(T^{-i} \alpha \vee \ldots \vee T^{-j} \alpha) = H(T^{-i-k} \alpha \vee \ldots \vee T^{-j-k} \alpha)$$

6. Prove the entropy inequalities (conditioning on sub-$\sigma$-algebras): see page 5 of Lectures 18-20 on course website).
7. True or False: 
\[ I_{a|\mathcal{B}} = E(I_{a}|\mathcal{B}) \]

8. Prove the two-sided generator theorem by modifying the proof of the one-sided generator theorem.

9. For an MPT \( T \) and positive integer \( n \) show that \( h(T^n) = nh(T) \).

10. Let \( \mathcal{T} \) be a \( \mathbb{Z}^d \) action and let \( \mathcal{T}^{-1} \) be the \( \mathbb{Z}^d \) action defined by \( \vec{i} \mapsto T^{-\vec{i}} \). Show that \( h(\mathcal{T}^{-1}) = h(\mathcal{T}) \).

11. Show that if an MP \( \mathbb{Z}^d \) action \( \mathcal{T} \) has a one-sided generator, then \( h(\mathcal{T}) = 0 \).

12. Compute the entropy of a rotation of the circle.

13. For each of the following, give one-sided generators and two-sided generators (whenever they make sense and exist):

   (a) The doubling map
   (b) The Baker’s transformation

14. Let \( \mathcal{T} \) be an MP \( \mathbb{Z}_+^d \) action with \( d \geq 2 \). Suppose that for some \( \vec{i} \in \mathbb{Z}_+^d \setminus \{0\} \), \( h(T^{\vec{i}}) < \infty \). Show that \( h(\mathcal{T}) = 0 \).