Geometry of convex sets

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Abstract

We present two classical theorems that give considerable geometric intuition into the notion of convexity. The statement of the first theorem is roughly that disjoint convex sets can be separated by a hyperplane, i.e. a level set of a linear functional. To accomplish this, we rely on the only general theorem we have for guaranteeing that linear functionals exist, namely the Hahn-Banach theorem. The statement of the second is essentially that a compact convex set can be recovered from its extreme points by taking convex combinations. This result uses the hyperplane separation theorem in an essential way, in concert with Zorn's lemma.

1 Separating convex sets with hyperplanes

The proofs in this section are from [1].

Theorem 1.1 (Hahn-Banach, separation version)**.** *Let X be a real topological vector space. Let A and B be disjoint convex sets in X with A open. Then there exist $f \in X^*$ and $\alpha \in \mathbb{R}$ such that $f|_A < \alpha \leq f|_B$.*

Proof. We are going to use the Hahn-Banach theorem, so we need a sublinear functional, which WLOG is the Minkowski functional of some convex set containing 0. A natural choice is $C = A - B - (a_0 - b_0)$ for some (arbitrary) $a_0 \in A$ and $b_0 \in B$. Observe that $A - a_0$ is a neighbourhood of 0, so C is, too, since $0 \in B - b_0$.

Let $z_0 = -(a_0 - b_0)$; with this notation, $C = A - B + z_0$. Observe that $z_0 \notin C$, since $0 \notin A - B$; indeed, if $0 \in A - B$, then $A \cap B \neq \emptyset$, contradicting the disjointness assumption. Therefore $\rho_C(z_0) \geq 1$. We define a linear functional $f_0 : \mathbb{R}z_0 \to \mathbb{R}$ by $f_0(sz_0) = s$. For $s \geq 0$, we have

$$f(sz_0) = sf(z_0) \leq s\rho_C(z_0) = \rho_C(sz_0) \tag{1}$$

and for $s \leq 0$ we have $f(sz_0) \leq 0 \leq \rho_C(sz_0)$, so ρ_C dominates f_0 on $\mathbb{R}z_0$. Thus f_0 extends to a linear functional $f : X \to \mathbb{R}$, which a priori might not be continuous, but is at least dominated by ρ_C. However, f is indeed continuous, since for any $\varepsilon > 0$, $\varepsilon C \cap (-\varepsilon C) \subset f^{-1}((-\varepsilon, \varepsilon))$.

We now need to find the claimed $\alpha \in \mathbb{R}$ such that the hyperplane $\{f = \alpha\}$ separates A and B. A reasonable guess is $\alpha = \sup f(A)$; we need to show that this choice of α has the required properties. First, observe that if x is a limit point of A with $f(x) = \alpha$, then in fact $x \in \partial A$. To see this, note that in that case $f(x) \in \partial f(A) = \overline{f(A)} \cap \overline{(f(A))^c}$, so $x \in \overline{A} \cap \overline{A^c} = \partial A$. Therefore $f|_A < \alpha$.

Furthermore, the same argument shows that $f|_C < 1$, since $f|_C \leq \rho_C|_C \leq 1$ and C is open. Thus, for any $a \in A$, $b \in B$, we have $f(a - b + z_0) < 1$, which shows that $f(a) < f(b)$ since $f(z_0) = 1$. By taking the supremum over $a \in A$, we see that $f|_B \geq \alpha$, concluding the proof. □

Different hypotheses yield an even stronger separation result, after a couple of lemmas:

Lemma 1.2. *Let X be a real TVS. For $f \in X^*$ and $C \subset X$ open and convex, $f(C)$ is an open interval.*

Proof. Since C is convex and f is linear, $f(C)$ is a convex subset of $\mathbb{R}$, thus an interval. If $f(C)$ is the whole real line, then it is certainly open, so we assume WLOG that $0 < \sup f(C) < \infty$. Suppose that $f(C)$ is closed on the right, so that $f(C) = (\inf f(C), \sup f(C)]$. Then there exists $x \in C \subset \overline{C}$ with $f(x) = \sup f(C)$, and for any $r > 1$, $r \sup f(C) = f(rx) \in f(C)^c$, so $rx \notin C$. But then there exists a sequence (r_n) with $r_n > 1$ and $r_n \to 1$, so $r_n x \to x$, meaning that $x \in \overline{C^c}$. Thus $x \in \overline{C} \cap \overline{C^c} = \partial C$. Since C is open, this contradicts the assumption that $x \in C$. □

Lemma 1.3. *Let A and B be convex sets. Then $A+B$ is also convex.*

Proof. Suppose that $a_1, a_2 \in A$, $b_1, b_2 \in B$, and $t \in [0,1]$. Then

$$t(a_1+b_1)+(1-t)(a_2+b_2) = [ta_1+(1-t)a_2]+[tb_1+(1-t)b_2] \in A+B \tag{2}$$

since each term is a convex combination of elements in a convex set. □

Your patience is now repaid:

Theorem 1.4. *Let X be a locally convex real TVS and A, B be disjoint closed convex sets in X with B compact. Then there exist $f \in X^*$, $\alpha \in \mathbb{R}$, and $\varepsilon > 0$ such that $f|_A < \alpha < \alpha + \varepsilon \le f|_B$.*

Proof. Since $B \subset A^c$ and A^c is open, there exists a neighbourhod U of 0 with $B+U \subset A^c$. Moreover, U is WLOG convex (since X is locally convex), so $B+U$ is also convex, as well as open. Thus $B+U$ and A are disjoint convex sets with $B+U$ open, so there exist $f \in X^*$ and $\alpha \in \mathbb{R}$ with $f|_A \le \alpha < f|_{B+U}$. By the earlier lemma, $f(B+U)$ is an open interval, of which $f(B)$ is a compact subinterval, so there exists $\varepsilon > 0$ with $\inf f(B) \ge \inf f(B+U) + \varepsilon = \alpha + \varepsilon$. □

2 Recovering a set from its extreme points

The material in this section is from [2]. For this section, unless stated otherwise, let X be a real, Hausdorff, locally convex topological vector space.

Definition 2.1. For any $x, y \in X$, we write $[x,y] = \{tx+(1-t)y : t \in [0,1]\}$ and $(x,y) = \{tx+(1-t)y : t \in (0,1)\}$.

Definition 2.2. Let $A \subset X$ be any set, and let $\emptyset \ne F \subset A$. We say that a closed set $F \subset A$ is a *face* of A if $[x,y] \subset F$ for any $x, y \in A$ with $(x,y) \cap F \ne \emptyset$.

Definition 2.3. An *extreme point* of a set $A \subset X$ is a point $x \in A$ such that, if $x \in (y,z) \subset A$, then $x = y = z$: that is, a face consisting of a single point. We write $\varepsilon(A)$ for the set of extreme points of A.

Proposition 2.1. *Let A be a set with a face $F \subset A$. Then a subset $B \subset F$ is a face of F if and only if it is a face of A.*

Proof. First suppose that $B \subset F$ is a face of A, that $[y,z] \subset F \subset A$, and that $x \in B \cap (y,z)$. Since B is a face of A, we have $x \in B$. Thus B is a face of F.

Conversely, suppose that $B \subset F$ is a face of F, that $[y,z] \subset A$, and that $x \in B \cap (y,z)$. Since $x \in B \subset F$ and F is a face, we must have $[y,z] \subset F$. Since B is a face of F and $[y,z] \subset F$ with $B \cap (y,z) \ne \emptyset$, we must have $[y,z] \subset B$. □

Corollary 2.1.1. *Let A be a set with a face $F \subset A$. Then $\varepsilon(F) = F \cap \varepsilon(A)$.*

Proposition 2.2. *Any compact convex subset of a locally convex space has extreme points. That is, let X be a locally convex space and let $A \subset X$ be a compact convex set. Then $\varepsilon(A) \ne \emptyset$.*

Proof. Consider the family $\mathcal{F}_{\alpha \alpha \in I}$ of faces of A, partially ordered by reverse inclusion. Let $\mathcal{C} = \mathcal{F}_{\alpha \alpha \in C}$ be a chain, and consider the intersection $F = \cap_{\alpha \in C} F_\alpha$. This is nonempty, by the intersection property for finite sets, and is moreover a face of A: indeed, if $[y,z] \subset A$ and $x \in F \cap (y,z)$, then for every $\alpha \in C$, we have $x \in F_\alpha \cap (y,z)$, so $[y,z] \subset F_\alpha$. Thus $[y,z] \subset F$.

Zorn's lemma therefore yields that A has some minimal face F. To see that F consists of a single point, we suppose that there exist $x, y \in F$ with $x \ne y$. Since X is locally convex and both $\{x\}$ and $\{y\}$ are closed and compact, the Hahn-Banach separation theorem implies that there exists a continuous linear functional $\ell \in X^*$ with $\ell(x) \ne \ell(y)$.

Let $\tilde{F} = \{z \in F : \ell(z) = \sup \ell(F)\}$; observe that since $\ell(x) \ne \ell(y)$, we cannot have both $x, y \in F$, so $\tilde{F} \subsetneq F$. Since F is compact (being a closed subset of the compact set A), $\tilde{F}$ is nonempty, and is clearly a face of F. By the "face-inheritance" proposition, $\tilde{F}$ is also a face of A, as well as a proper subset of F, contradicting the minimality of F. □

Definition 2.4. Let X be a vector space and let $A \subset X$ be a set. The *convex hull* of A, written $\mathrm{ch}(A)$, is the set of convex combinations of points in A. Symbolically, $\mathrm{ch}(A) = \cup_{x,y \in A}[x, y]$.

Observe that every convex set at least contains the convex hull of its extreme points. Moreover, we have the following rather satisfying result:

Theorem 2.3 (Krein-Milman)**.** *Let A be a compact convex subset of a locally convex vector space. Then A is equal to the closure of the convex hull of its extreme points. That is, $A = \overline{\mathrm{ch}(\varepsilon(A))}$.*

Proof. By the above remark, we certainly have $\mathrm{ch}(\varepsilon(A)) \subseteq A$; since A is compact, hence closed, we in fact have $\overline{\mathrm{ch}(\varepsilon(A))} \subseteq A$. Our goal is equality, which we pursue by assuming that $\overline{\mathrm{ch}(\varepsilon(A))} \subsetneq A$ and seeking a contradiction.

Let $B = \overline{\mathrm{ch}(\varepsilon(A))}$ and suppose there exists some $x_0 \in A \setminus B$. By the Hahn-Banach theorem for compact convex sets, applied to B and $\{x_0\}$, there exists $\ell \in X^*$ with $\ell(x_0) > \sup \ell(B)$. Let $F = \{x \in A : \ell(x) = \sup \ell(A)\}$, which is clearly a face of A. Moreover, for any $x \in F$, we have $\ell(x) \geq \ell(x_0) > \sup \ell(B)$, so we have $F \cap B = \emptyset$.

Finally, since F is compact and convex, F has an extreme point y_0, which is also an extreme point of A, and thus contained in $\varepsilon(A) \subset B$. This contradicts the disjointness of F and B. Thus $B = A$, as claimed. $\square$

References

[1] Brian Bockelman. "Functional Analysis Notes: Hahn-Banach separation theorems." Personal webpage at the University of Nebraska-Lincoln, last updated 2006. Available at http://www.math.unl.edu/ s-bbockel1/929/node2.html.

[2] Barry Simon. *Convexity: An Analytic Viewpoint.* Cambridge, 2011.