Math 421W2013T2 Solutions

2a.

$$d(x, y) = \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n(\max(|x_n|, 1))(\max(|y_n|, 1))}$$

b. A function in L^2 but not in L^3: $f(x) = x^{-1/3}$

c. Done in class. Recall example of derivative map from $C^1(\Omega)$, with sup norm, to $C(\Omega)$, with sup norm. As shown in class this map is a closed, i.e., graph is closed, linear map that is not continuous (this does not contradict closed graph theorem because C^1 is not Banach). It is unbounded because the ratio of sup of —derivarive— to sup of —original function— can be unbounded – we also did this in class.

3. N/A

4a. State the result.

b. Show that $I(f)$ is a positive linear functional. Apply Riesz.

5. Not responsible for nets. In case of sequences, just separate two distinct alleged limit points by open sets.

6. True. $f_n \overset{wk*}{\to} f$ means that for all $x \in X$, $f_n(x) \to f(x)$. But then for each x, $\{f_n(x)\}_n$ is bounded in K and thus by UBP, $\{f_n\}$ is bounded in the operator norm on X^*.

7. Spectral Theorem N/A

8. Application of Open Mapping Theorem done in class

9. The set c of all convergent sequences is a subspace of ℓ^∞.
For $x \in c$, let $g(x) := \lim_n g(x_n)$. Clearly g is a linear functional on c.
We claim that $p(x) := \limsup x_n$ is a sublinear fucntional on ℓ^∞.
For $\lambda \geq 0$, clearly $p(\lambda x) := \limsup \lambda x_n = \lambda \limsup x_n$.
And

$$p(x+y) = \limsup x_n + y_n = \lim(\sup(x_n + y_n)) \leq \limsup x_n + \limsup y_n = p(x) + p(y)$$

Since $g \leq p$ on c, g extends to all of ℓ^∞ s.t. $g \leq p$.
So, for all $x \in \ell^\infty$, $g(x) \leq \limsup x_n$.
And

$$-g(x) = g(-x) \leq \limsup -x_n = -\liminf x_n$$

So, $g(x) \geq \liminf x_n$.

Finally,

$$-\sup|x_n| = \inf -|x_n| \leq \inf x_n \leq \liminf x_n \leq g(x) \leq \limsup x_n \leq \sup x_n \leq \sup|x_n|$$

It follows that

$$|g(x)| \leq \sup|x_n| = ||x||_\infty$$

So, $||g|| \leq 1$. But since $g(1,1,1,\ldots) = 1$, we have $||g|| = 1$.