Math 421W2005T2 Solutions

1.b Nhhd base at $f$:
\[
\left\{ \cap_{i=1}^m p_{\alpha_i}^{-1}(U_{\alpha_i}) \right\}
\]
where each $U_{\alpha_i}$ is open in $Y$ and contains $f(\alpha_i)$; here, the $p_{\alpha}$ are projection maps.

c. In fact, pointwise convergence is equivalent to convergence in the product topology. In particular, given pointwise convergence to $f$, for any $U$ above in the nbhd base at $f$ for suff. large $N$ and all $n \geq N$, $f_n \in U$ because $f(\alpha_i) \in U_{\alpha_i}$ for $i = 1, \ldots, m$.

d. No. Cantor diagonalization argument gives counterexample.

Let $f_n$ be sequence s.t. $f_n(x) = n$-th binary digit of $x$.

Then $f_n$ has not convergent subsequence because a subsequence $f_{n_k}$ is convergent iff for all $x$ $f_{n_k}(x)$ is eventually constant (convergence in product space = pointwise or coordinate wise convergence) but for any subsequence $n_k$ just choose an $x \in [0,1]$ s.t. $x_{n_k} = 1$ if $k$ is odd and 0 if $k$ is even.

This space is compact but not sequentially compact.

2. a, b, statements of results in the text.

c. done in class, Folland - Theorem 5.8b.

d. Choose $f$ s.t. $||f|| = 1$ and $f(x) = ||x||$.

Since $x_n \overset{w}{\rightarrow} x$, $f(x_n) \rightarrow f(x) = ||x||$. Thus,
\[
||x|| = f(x) = |f(x)| = \lim_n |f(x_n)| \leq \liminf_n ||f|| ||x_n|| = \liminf_n ||x_n||
\]

e. $||x_n - x||^2 = ||x_n||^2 - 2\langle x_n, x \rangle \rightarrow 2||x||^2 - 2\langle x, x \rangle = 0$.

f. Yes. In $\ell^2$, $e_n \overset{w}{\rightarrow} 0$ because for every $y \in \ell^2$, $\langle e_n, y \rangle \rightarrow 0$. Yet $||e_n|| = 1$ and $||0|| = 0$.

3a. No. If cts. functions were dense in $L^\infty$ (with the sup norm then so would the polynomials with rational coefficients (by Stone-Weirstrass) and then $L^\infty$ would be separable. But we proved in class that it is not separable.

b. State defn.

c. Since the inner product of $H$ is a continuous function, if $f \perp D$ for a dense set $D$, then $f \perp H$. But then $\langle f, f \rangle = 0$. 

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d. To be a complete o.n. set means that the closed linear span of the set is the entire space. Since compactly supported (in $(0,1)$) cts. functions are dense it suffices to show that the closed linear span of the o.n. set contains \( C([a, b]) \) for all \( 0 < a < b < 1 \). Apply Stone-Weirstrass. The set includes the constant functions and \( e_1 \) already separates points. So it is a complete o.n. set.

4. a. norm, strong and weak.
   b. Fix \( x \). For all \( f \in X^* \), \( \hat{T}_n x(f) = f(T_n x) \to f(T x) \). Thus, for fixed \( f \), \( \{||\hat{T}_n x(f)||\}_n \) is bounded. By Uniform Boundedness Principle, for fixed \( x \), \( \{||T_n x||\}_n \) is bounded.
   
   But since \( y \mapsto \hat{y} \) is an isometric isomorphism, \( ||\hat{T}_n x|| = ||T_n x|| \). So, for fixed \( x \), \( \{||T_n x||\}_n \) is bounded. By Uniform Boundedness Principle, \( \{||T_n||\} \) is bounded by some \( M > 0 \).
   
   For given \( x \in X \), there exists \( f \in X^* \) s.t. \( ||f|| = 1 \) and \( f(T x) = ||T x|| \).
   
   Thus,
   
   \[
   ||T x|| = |f(T x)| = \lim |f(T_n x)| \leq \sup_n ||f|| ||T_n x||
   \]
   \[
   = \sup_n ||T_n x|| \leq \sup_n ||T_n|| ||x|| \leq M ||x||
   \]
   
   So, \( T \) is a BLT.

5. N/A