Math 421W2005T2 Solutions
1.b Nhhd base at $f$ :

$$
\left\{\cap_{i=1}^{m} p_{\alpha_{i}}^{-1}\left(U_{\alpha_{i}}\right)\right\}
$$

where each $U_{\alpha_{i}}$ is open in $Y$ and contains $f\left(\alpha_{i}\right)$; here, the $p_{\alpha}$ are projection maps.
c. In fact, pointwise convergence is equivalent to convergence in the product topology. In particular, given pointwise convergence to $f$, for any $U$ above in the nbhd base at $f$ for suff. large $N$ and all $n \geq N, f_{n} \in U$ becauses $f\left(\alpha_{i}\right) \in U_{\alpha_{i}}$ for $i=1, \ldots, m$.
d. No. Cantor disagonolization argument gives counterexample.

Let $f_{n}$ be sequence s.t. $f_{n}(x)=n$-th binary digit of $x$.
Then $f_{n}$ has not convergent subsequence vvv ff because a subsequence $f_{n_{k}}$ is convergent iff for all $x f_{n_{k}}(x)$ is eventually constant (convergence in product space $=$ pointwise or coordinate wise convegence) but for any subsequence $n_{k}$ just choose an $x \in[0,1]$ s.t. $x_{n_{k}}=1$ if $k$ is odd and 0 if $k$ is even.

This space is compact but not sequentially compact.
2. $\mathrm{a}, \mathrm{b}$, statements of results in the text.
c. done in class, Folland - Theorem 5.8b.
d. Choose $f$ s.t. $\|f\|=1$ and $f(x)=\|x\|$.

Since $x_{n} \xrightarrow{w} x, f\left(x_{n}\right) \rightarrow f(x)=\|x\|$. Thus,

$$
\|x\|=f(x)=|f(x)|=\lim _{n}\left|f\left(x_{n}\right)\right| \leq \liminf _{n}\|f\|\left\|x_{n}\right\|=\liminf _{n}\left\|x_{n}\right\|
$$

e.

$$
\left\|x_{n}-x\right\|^{2}=\left\|x_{n}\right\|^{2}-2\left\langle x_{n}, x\right\rangle \rightarrow 2\|x\|^{2}-2\langle x, x\rangle=0 .
$$

f. Yes. In $\ell^{2}, e_{n} \xrightarrow{w} 0$ because for every $y \in \ell^{2},\left\langle e_{n}, y\right\rangle \rightarrow 0$. Yet $\left\|e_{n}\right\|=1$ and $\|0\|=0$.

3a. No. If cts. functions were dense in $L^{\infty}$ (with the sup norm then so would the polynomials with rational coefficients (by Stone-Weirstrass) and then $L^{\infty}$ would be separable. But we proved in class that it is not separable.
b. State defn.
c. Since the inner product of $H$ is a continuous function, if $f \perp D$ for a dense set $D$, then $f \perp H$. But then $\langle f, f\rangle=0$.
d. To be a complete o.n. set means that the closed linear span of the set is the entire space. Since compactly supported (in (0,1)) cts. functions are dense it suffices to show that the the closed linear spane of the o.n. set contains $C([a, b])$ for all $0<a<b<1$. Apply Stone-Weirstrass. The set includes the constant functions and $e_{1}$ already separates points. So it is a complete o.n. set.
4. a. norm, strong and weak.
b.

Fix $x$. For all $f \in X^{*}, \widehat{T_{n} x}(f)=f\left(T_{n} x\right) \rightarrow f(T x)$. Thus, for fixed $f$, $\left\{\left\|\widehat{T_{n} x}(f)\right\|\right\}_{n}$ is bounded. By Uniform Boundedness Principle, for fixed $x$, $\left\{\left\|\widehat{T_{n} x}\right\|\right\}_{n}$ is bounded.

But since $y \mapsto \hat{y}$ is an isometric isomorphism, $\left\|\widehat{T_{n} x}\right\|=\left\|T_{n} x\right\|$. So, for fixed $x,\left\{\left\|T_{n} x\right\|\right\}_{n}$ is bounded. By Uniform Boundedness Principle, $\left\{\left\|T_{n}\right\|\right\}$ is bounded by some $M>0$.

For given $x \in X$, there exists $f \in X^{*}$ s.t. $\|f\|=1$ and $f(T x)=\|T x\|$. Thus,

$$
\begin{aligned}
\|T x\| & =|f(T x)|=\lim \left|f\left(T_{n} x\right)\right| \leq \sup _{n}\|f\|\left\|T_{n} x\right\| \\
& =\sup _{n}\left\|T_{n} x\right\| \leq \sup _{n}\left\|T_{n}\right\|\|x\| \leq M\|x\|
\end{aligned}
$$

$\mathrm{So}, T$ is a BLT.
5. N/A

