Basis: (Infinite set)
, Spanning
, Lin. Indep: for any finite subset $x_1, \dots, x_k$ $\sum a_i x_i = 0 \Rightarrow$ all $a_i = 0$

Lecture 9:

Note: We proved that the unit ball is not compact in an infinite dimensional NVS. It is an easy corollary that any ball is not compact, essentially because the vector space operations, addition and scalar multiplication, are continuous.

basis

Theorem: Every non-zero vector space X has a Hamel basis. More generally, if W is a subspace of X any Hamel basis for W can be extended to a Hamel basis for X.

POSET

Proof of Theorem: Let S be the collection of linearly independent subsets of X that contain B, a basis of W, ordered by inclusion. Let C be a totally ordered subset of S, i.e., for any pair of elements of C, one is a subset of the other. Note that any finite collection of elements of C can be ordered $C_1 \subset C_2 \subset \ldots \subset C_k$ and in particular $C_k = \cup_{i=1}^{k} C_i$.

Then $U := \cup_{\{c \in C\}} c$ is an upper bound for C and indeed U is linearly independent because any finite subset of U is contained in the union of finitely many elements of C, $C_1 \subset C_2 \subset \ldots \subset C_k$, therefore in a single element C_k.

Zorn gives us a maximal linearly independent subset M, i.e., if $M \subseteq T$ and T is linearly independent, then $T = M$.

We claim that M is also spanning. If not, then there is some element $x \in X$ that is not a linear combination of elements of M. Then $M \cup \{x\}$ is a linearly independent proper super-set of M, contradicting maximality of M. $\sum_{i=1}^{k} a_i x_i + ax = 0$

And clearly M contains B. Thus, M is a Hamel basis for X that contains B.

To show that X does indeed have a basis, let B be any single nonzero element x, which is clearly a basis for the subspace Kx, and apply the foregoing argument. □

Fact: Any two Hamel bases of a vector space X have the same cardinality, and this common cardinality is called the *dimension* of X.

Later we will show that the dimension of any Banach space is either finite or uncountable, equivalently cannot be countably infinite. This is a consequence of the Baire Category Theorem.

So, Hamel bases are not so useful for Banach spaces. Later, we will consider more useful bases, called Schauder bases, for Banach spaces.

Hahn-Banach Theorem

not BLF

Algebraic HB Theorem:

Let W be a subspace of X, vector space. Let $f : W \to K$ be a linear functional. Then f can be extended to a linear functional F on X, i.e., $F|_W = f$.

Note: This is purely linear algebra (do not need a norm or metric or even topology).

Proof: Extend a Hamel basis on W to a Hamel basis on X and define F to be 0 on the new basis elements (or anything else you want). □

Note that this uses Zorn's lemma because of proof of existence of Hamel bases.

Defn: A *sublinear functional* $p : X \to \mathbb{R}$ is a mapping which satisfies $p(x + y) \leq p(x) + p(y)$ and $p(\lambda x) = \lambda p(x)$ for all $x, y \in X$ and $\lambda \geq 0$.

Example: Let $p(x) := c||x||$ where $c > 0$ is constant. Observe that p is a sublinear functional:

$$p(x + y) = c||x + y|| \leq c||x|| + c||y|| = p(x) + p(y)$$

not BLF

$$p(\lambda x) = c||\lambda x|| = \lambda c||x|| = \lambda p(x), \;\; \lambda \geq 0$$

Real HB Theorem: Let W be a subspace of X, a real vector space. Let $f : W \to \mathbb{R}$ be a linear functional. Let $p : X \to \mathbb{R}$ be a sublinear functional such that for all $x \in W$, $f(x) \leq p(x)$. Then f can be extended to a linear functional F on X s.t. $F(x) \leq p(x)$ for all $x \in X$ and $F|_W = f$.

Before we give the proof, we establish the most important application.

Corollary: Let W be a subspace of X, a real NVS. Let $f : W \to \mathbb{R}$ be a BLF. Then f can be extended to a BLF F on X s.t. $||F|| = ||f||$.

Proof: Apply real HB with $p(x) = ||f||\;||x||$.

Since f is a BLF, for all $w \in W$,

$$\underbrace{f(w) \leq |f(w)| \leq ||f||\;||w|| = p(w)}_{\leq}$$

By real HB theorem, there is a linear extension F of f to X s.t. for all $x \in X$

$$F(x) \leq ||f||\;||x||$$

We claim that in fact

$$|F(x)| \leq ||f||\;||x||$$

If $F(x) \geq 0$, already done.

If $F(x) < 0$, then by linearity of F,

$$|F(x)| = -F(x) = F(-x) \leq ||f||\;||-x|| = ||f||\;||x||$$

So F is a BLF on X s.t. $||F|| \leq ||f||$.

But in fact $||F|| = ||f||$ since $||f||$ already equals $\sup_{w \in W:\; w \neq 0} \frac{|F(w)|}{||w||}$.

$\square$

Lecture 10:

Re-posted HW2 on Tuesday; clarified defn. of $K_0(\Omega)$ in problem 8.

Recall:

Real HB Theorem: Let W be a subspace of X, a real vector space. Let $f : W \to \mathbb{R}$ be a linear functional. Let $p : X \to \mathbb{R}$ be a sublinear functional such that for all $x \in W$, $f(x) \leq p(x)$. Then f can be extended to a linear functional F on X s.t. $F(x) \leq p(x)$ for all $x \in X$ and $F|_W = f$.

Proof of real HB Theorem:

Step 1: One-dimensional extensions: Given a linear functional f on a subspace W and sublinear functional p on X that satisfies $f(w) \leq p(w)$ for all $w \in W$, and $y \in X \setminus W$, find a linear extension F of f to $U = W + \mathbb{R}y$ such that $F(u) \leq p(u)$ for all $u \in U$.

Since an extension F is linear, it is completely determined by its value on y. We will now derive necessary and sufficient conditions on the value $F(y) = \alpha$ that will allow such an extension.

We need that for all real λ

$$f(w) + \lambda\alpha = F(w + \lambda y) \leq p(w + \lambda y) \tag{2}$$

Of course, this automatically holds for $\lambda = 0$.

For fixed $\lambda > 0$, (2) is equivalent to:

$$\alpha \leq (1/\lambda)(p(w + \lambda y) - f(w)) = p(w/\lambda + y) - f(w/\lambda)$$

Since $w/\lambda \in W$ iff $w \in W$, this holds iff

Reversible

$$\alpha \leq \inf_{w \in W} p(w + y) - f(w) \tag{3}$$

For fixed $\lambda < 0$, (2) is equivalent to:

$$\alpha \geq (1/\lambda)(p(w + \lambda y) - f(w)) = -p(-w/\lambda - y) + f(-w/\lambda) \tag{4}$$

Since $-w/\lambda \in W$ iff $w \in W$, this holds iff

$$\alpha \geq \sup_{w \in W} -p(w-y) + f(w)$$

Thus, a necessary and sufficient condition on α such that the extension works is:

$$\sup_{w \in W} -p(w-y) + f(w) \leq \alpha \leq \inf_{w \in W} p(w+y) - f(w)$$

But the existence of such an α is just the condition that the LHS is $\leq$ than the RHS of the preceding line.

This is equiavlent to the condition that for all $w_1, w_2 \in W$,

$$-p(w_1 - y) + f(w_1) \leq p(w_2 + y) - f(w_2)$$

which is in turn equivalent to

$$f(w_1) + f(w_2) \leq p(w_2 + y) + p(w_1 - y)$$

But this is true since

$$f(w_1) + f(w_2) = f(w_1 + w_2) \leq p(w_1 + w_2) \leq p(w_2 + y) + p(w_1 - y)$$

by subadditivity of p. $\square$

Step 2: Order the set $\mathcal{S}$ of pairs (W', f') of subspaces and linear functionals f' on W' s.t.

$$W \subset W', \ \ f'|_W = f, \ \ f' \leq p \text{ on } W',$$

by inclusion:

$$(W', f') \leq (W'', f'') \text{ if } W' \subset W'', \ \ f''|_{W'} = f'.$$

Apply Zorn:

We Claim that every totally ordered subset $\mathcal{C}$ has an upper bound (U, G).

Let

$$U := \cup_{(W',f')\in \mathcal{C}} W'.$$

Let $G : U \to \mathbb{R}$ be defined by $G(u) = f'(u)$ for any $(W', f') \in \mathcal{C}$ s.t. $u \in W'$.

Proof of Claim: we must verify:

1. $(U, G) \in \mathcal{S}$
   (a) U is a subspace
   (b) G is well-defined
   (c) $W \subseteq U$
   (d) $G|_W = f$
   (e) $G \leq p$
2. (U,G) is an upper bound of $\mathcal{C}$: for all $(W', f') \in \mathcal{C}$,
   (a) $W' \subseteq U$
   (b) $G|_{W'} = f'$

1a: If $x, y \in U$ and $a, b \in \mathbb{R}$, then $x \in W'$, $y \in W''$ and so because of the total ordering, one of W', W'' is contained in the other, say $W' \subset W''$, and so x, y belong to W'' and so $ax + by \in W'' \subset U$.

1b: again because of the total ordering: if $u \in W' \cap W''$, then say $W' \subset W''$, and then $f'(u) = f''(u)$.

1c: because for each $(W', f') \in \mathcal{C}$, $W \subseteq W'$

1d: $G|_W = f'|_W = f$

1e: since $G|_{W'} = f' \leq p$

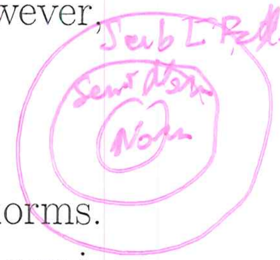
2a,b: follow from defn. of (U, G)

By Zorn, there is a maximal element (M, F). If $M \neq X$, then let $y \in X \setminus M$ and apply Step 1 to find a one-dimensional extension of F, contrary to maximality of (M, F). $\square$

A *semi-norm* is a function $x \mapsto ||x|| \geq 0$ on a vector space that satisfies

- Homogeneity $\cancel{p(\lambda x) = |\lambda| p(x)}$ $||\lambda x|| = |\lambda| \cdot ||x||$
- Subadditivity. $\cancel{p(x+y) \leq p(x) + p(}$ $||x+y|| \leq ||x|| + ||y||$

Note that it follows from homogeneity that $||0|| = 0$. However, $||x||$ can be 0 without $x = 0$.



Note: Norm $\Rightarrow$ Semi-norm $\Rightarrow$ Sublinear functional.

So, the real HB Theorem applies to semi-norms as well as norms. Some applications require sublinear functionals that are not semi-norms.

Main example of semi-norm that is not a norm:

Let X be a vector space ℓ a linear functional, $X \to K$. Then $||x|| := |\ell(x)|$ is a semi-norm, and it is not a norm iff $Ker(\ell) \neq \{0\}$.

(So, "most" linear fuctionals do not define norms).

Proof: Clearly, $||x|| \geq 0$ and $||x|| = 0$ iff $x \in Ker(\ell)$.

By linearity,

$|\lambda| |\ell(x)| = |\lambda| ||x||$

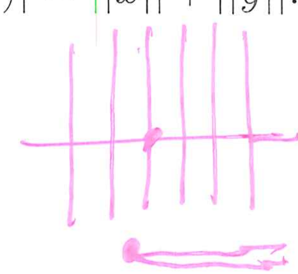
$$||\lambda x|| = |\ell(\lambda x)| = \cancel{|\lambda||\lambda(x)| = |\lambda|||x||}$$

By linearity

$$||x + y|| = |\ell(x + y)| = |\ell(x) + \ell(y)| \leq |\ell(x)| + |\ell(y)| = ||x|| + ||y||.$$

$\square$

More specific example: on R^2, $||(x, y)|| = |x|$.



Return HW 1

Anders: +1 Simon +2

~~More specific example: on R^2, $||(x,y)|| = |x|$.~~

Ben +1 Adam M. +2

Lecture 11:

Complex Hahn-Banach Theorem.

Defn: Let X be a complex vector space. ($\therefore$ also an $\mathbb{R}$-vector space)

Dylan +1

A $\mathbb{C}$-*linear functional* is an ordinary linear functional $h : X \to \mathbb{C}$, i.e.,

– $h(x+y) = h(x) + h(y)$ and

– $h(\lambda x) = \lambda h(x)$ for all $\lambda \in \mathbb{C}$.

A $\mathbb{R}$-*linear functional* is a linear mapping $f : X \to \mathbb{R}$ that is linear over $\mathbb{R}$.

– $f(x+y) = f(x) + f(y)$ and

– $f(rx) = rf(x)$ for all $r \in \mathbb{R}$. $\mathbb{C}$-linear $\Rightarrow$ $\mathbb{R}$-linear

Example: $X = \mathbb{C}^2$ and $f(w,z) = Re(w)$, equivalently

$$f((a+ib, c+id)) = a.$$

f is $\mathbb{R}$-linear:

Clearly, f preserves addition. And

$$f(r((a+ib,c+id))) = f((ra+irb,rc+ird)) = ra = rf((a+ib,c+id))$$

f is not $\mathbb{C}$-linear:

$f(i(a+ib, c+id)) = -b.$
$if((a+ib, c+id)) = ia.$

How are real and complex linear functionals related?
Prop 1: Let X be a complex vector space.

a. Let f be a $\mathbb{R}$-linear functional.

Then $f_C(x) := f(x) - if(ix)$ is a $\mathbb{C}$-linear functional, the complexification of f.

b. Let h be a $\mathbb{C}$-linear functional and let $f = \Re h$.

Then f is a $\mathbb{R}$-linear functional and $h(x) = f_{\mathbb{C}}(x)$.

$$\begin{array}{ccc} \mathbb{R} & & \mathbb{C} \\ f = \Re h & \Leftrightarrow & h = f_{\mathbb{C}} \end{array}$$

The real part uniquely determines the entire functional.

Proof:

a. For $r \in \mathbb{R}$,
$f_{\mathbb{C}}(rx) = f(rx) - if(irx) = rf(x) - rif(ix) = rf_{\mathbb{C}}(x)$
So, $f_{\mathbb{C}}$ is $\mathbb{R}$-linear.

$f_{\mathbb{C}}(ix) = f(ix) - if(-x) = f(ix) + if(x) = i(f(x) - if(ix)) = if_{\mathbb{C}}(x)$
Since $f_{\mathbb{C}}$ is $\mathbb{R}$-linear and "i-linear," it is $\mathbb{C}$-linear because

$$f_{\mathbb{C}}((a+bi)x) = f_{\mathbb{C}}(ax) + f_{\mathbb{C}}(bix) = af_{\mathbb{C}}(x) + bif_{\mathbb{C}}(x) = (a+bi)f_{\mathbb{C}}(x)$$

b. Write $h(x) = f(x) + ig(x)$.

For all $r \in \mathbb{R}$,

$$f(rx) + ig(rx) = h(rx) = rh(x) = r(f(x) + ig(x)) = rf(x) + rig(x)$$

and so comparing real parts, we see that $f(rx) = rf(x)$ and so f is $\mathbb{R}$-linear. And

$$f(ix) + ig(ix) = h(ix) = ih(x) = i(f(x) + ig(x)) = if(x) - g(x)$$

Comparing real parts, we see that

$$g(x) = -f(ix)$$

and so $h(x) = f(x) - if(ix) = f_{\mathbb{C}}(x)$. $\square$

K-

Defn of K-semi-norm: a function $x \mapsto p(x) \geq 0$ on a vector space that satisfies

- K-Homogeneity: for all $x \in X$ and $\lambda \in K$, $p(\lambda x) = |\lambda|\, p(x)$
- Subadditivity: for all $x, y \in X$, $p(x+y) \leq p(x) + p(y)$.

Prop 2: Let f be a $\mathbb{R}$-linear functional.

a. Let $p(x)$ be a $\mathbb{C}$-seminorm on X. Then for all $x \in X$, $|f(x)| \leq p(x)$ iff for all $x \in X$, $|f_{\mathbb{C}}(x)| \leq p(x)$.

b. Let $||\cdot||$ be a norm on X. Then $||f_{\mathbb{C}}|| = ||f||$.

Proof:

a. If:

$$|f(x)| = |(\Re f_{\mathbb{C}})(x)| \leq |f_{\mathbb{C}}(x)| \leq p(x)$$

Only If: Suppose that for all $x \in X$, $|f(x)| \leq p(x)$.

For $x \in X$, for some $\theta_x \in \mathbb{C}$ s.t. $|\theta_x| = 1$ and

$$|f_{\mathbb{C}}(x)| = \theta_x f_{\mathbb{C}}(x).$$

because it is real ✓

& ≥ 0

Then

$$|f_{\mathbb{C}}(x)| = \theta_x f_{\mathbb{C}}(x) = f_{\mathbb{C}}(\theta_x x) = \Re f_{\mathbb{C}}(\theta_x x)$$
$$= |f(\theta_x x)| \leq p(\theta_x x) = |\theta_x| p(x) = p(x).$$

b. $||f|| \leq ||f_{\mathbb{C}}||$:

& ≥ 0

follows from:

$$|f(x)| \leq |f_{\mathbb{C}}|(x).$$

$||f_{\mathbb{C}}|| \leq ||f||$:

Let θ_x be as in proof of part a, i.e. $|\theta_x| = 1$ and

$$|f_{\mathbb{C}}(x)| = \theta_x f_{\mathbb{C}}(x)$$

Then as above

$$|f_{\mathbb{C}}(x)| = |f(\theta_x x)|$$

So

$$||f_{\mathbb{C}}|| = \sup_{x\in X:\ ||x||=1} |f_{\mathbb{C}}(x)| = \sup_{x\in X:\ ||x||=1} |f(\theta_x x)| \le \sup_{y\in X:\ ||y||=1} |f(y)| = ||f|| \square$$

Note: in foregoing, we really needed $p(x)$ to be a $\mathbb{C}$-semi-norm rather than a sublinear functional because we used $p(\theta_x x) = |\theta_x| p(x)$ for a complex θ_x.

Complex Hahn-Banach Theorem: Let X be a complex vector space and $p(x)$ a $\mathbb{C}$-semi-norm on X.

Let W be a complex subspace of X.

Let h be $\mathbb{C}$-linear-functional on W s.t, for all $w \in W$, $|h(w)| \le p(w)$.

Then there is a $\mathbb{C}$-linear-functional H on X s.t. $H|_W = h$ and for all $x \in X$, $|H(x)| \le p(x)$.

Proof:

$$\begin{array}{cccc}
 & \mathbb{R} & & \mathbb{C} \\
X & F & \rightarrow & H = F_{\mathbb{C}} \\
 & \uparrow & & \\
W & f = \Re h & \leftarrow & h
\end{array}$$

Let $f = \Re h$, and so by Prop 1b, $\boxed{h = f_{\mathbb{C}}}$.

Then f is a $\mathbb{R}$-linear functional on W s.t. for all $w \in W$, $f(w) \le |f(w)| \le |h(x)| \le p(w)$.

Since p is a $\mathbb{C}$-semi-norm, it is a $\mathbb{R}$-semi-norm and hence a semi-linear functional.

By real HB, there is a $\mathbb{R}$-linear functional F on X s.t. $F|_W = f$ and for all $x \in X$, $F(x) \le p(x)$.