Card Access

Lecture 6:

Recall:

Corollary: If Y is closed subset of a metrix space X and $x \notin Y$, then $d(x, Y) > 0$.

Proposition: Let X be a NVS and Y a proper closed subspace of X. Let

$$d_{max} := \sup_{x \in X:\ ||x||=1} d(x, Y)$$

$d(x, Y) = \inf_{y \in Y} ||x - y||$

Then $d_{max} = 1$.

Proof: $d_{\max} \leq 1$:
$d(x, Y) \leq d(x, 0) = ||x|| = 1$.

$d_{\max} \geq 1$:
Let $z \in X \setminus Y$ and $d = d(z, Y) > 0$. Let $a < 1$.

Choose $y_0 \in Y$ s.t.

$$d \leq ||z - y_0|| \leq d/a$$

and let $x = \frac{z - y_0}{||z - y_0||}$. Clearly $||x|| = 1$. For any $y \in Y$,

$$||x - y|| = ||\frac{z - y_0}{||z - y_0||} - y|| = \frac{||z - y_0 - y||z - y_0||||}{||z - y_0||} \geq d/(d/a) = a$$

since $y_0 + y||z - y_0|| \in Y$.

Thus, $d(x, Y) \geq a$. Thus, $d_{\max} \geq 1$. $\square$

Proposition: Any finite dimensional subspace of an NVS is a closed subspace.

Proof: The subspace Y inherits the norm from the ambient space X. Since Y is a finite dimensional NVS, its norm is equivalent to the euclidean norm and thus Y is complete. But any complete subset of an NVS is closed (as proven in HW1, #1a; note that in that exercise

you don't need to assume that the ambient metric space is complete).
□

Theorem: The unit sphere S_1 in any infinite-dimensional NVS X (in particular, an infinite-dimensional Banach space) is *not* compact.

Proof: We will use the preceding proposition but only that $d_{max} > 1/2$, i.e. for any closed proper subspace Y of X, there exists some $x \in S_1$ s.t. $d(x, Y) > 1/2$.

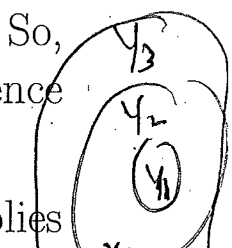
Inductively construct a sequence of n-dimensional subspaces Y_n. Start with a 1-dimensional subspace Y_1.

Inductively given an n-dimensional subspace Y_n choose $x_n \in S_1$ s.t. $d(x_n, Y_n) > 1/2$.

Set $Y_{n+1} = \text{span}(Y_n, x_n)$, an $(n+1)$-dimensional subspace.

This is possible since X is infinite dimensional and so Y_n is always a proper closed subspace.

Then for all m ~~n~~ and $n < m$ ~~$m \geq n$~~, $x_n \in Y_m$ and so $||x_m - x_n|| > 1/2$. So, x_n is a sequence of points on unit sphere, with no Cauchy subsequence and therefore no convergent subsequence. □.



Note: we are not using completeness, only that convergent implies Cauchy.

Corollary: The closed unit ball B_1 in any infinite-dimensional NVS (in particular, an infinite-dimensional Banach space) is not compact.

Proof: If unit ball were compact, then any closed subset of unit ball would be compact. But unit sphere is closed and not compact.
□

In particular, in an NVS a set can be closed and bounded but not compact.

However, later we will find another meaningful topology in which the unit ball in many Banach spaces is compact.

This is important for constructing certain probability measures and invariant measures in ergodic theory.

Continuous linear maps

For NVS X and Y, a linear transformation $T : X \to Y$ is *bounded (BLT)* if there exists $C > 0$ s.t. for all $x \in X$, $||Tx|| \leq C||x||$.

i.e., T expands distances from origin by at most some uniform constant C.

Theorem: Let $T : X \to Y$ be a linear transformation from one vector space X to another Y. The following are equivalent.

1. T is a BLT
2. T is uniformly continuous
3. T is continuous
4. T is continuous at 0

Proof:

1 implies 2: Assume $||Tx|| \leq C||x||$ for all x. Then for all $\epsilon > 0$, if $||x - y|| < \delta := \epsilon/C$, then

$$||Tx - Ty|| = ||T(x - y)|| \leq C||x - y|| < C\epsilon/C = \epsilon.$$

So, T is uniformly continuous.

2 implies 3 implies 4 : obvious

4 implies 1: By definition of continuity at 0, with $\epsilon = 1$, there exists $\delta > 0$ s.t. if $||x|| \leq \delta$, then $||Tx|| \leq 1$. Then for all $x \neq 0$, let $y = \delta \frac{x}{||x||}$. Then,

$$\frac{\delta}{||x||}||T(x)|| = ||T(y)|| \leq 1.$$

Thus,

$$||T(x)|| \leq (1/\delta)||x||$$

This inequality also holdes for $x = 0$. Thus, letting $C = 1/\delta$, we get for all $x \neq 0$,

$$||T(x)|| \leq C||x||$$

Thus, T is a BLT.. $\square$

2, 3, 4

Conditions ~~1,2~~ and ~~3~~ are, of course, continuity conditions.

Conditions ~~4~~ 1 is a boundedness condition.

Note that BLT does not mean that T is bounded in the usual sense that the image of T is bounded. In fact, the only linear transformation bounded in that sense is the 0 transformation.

The BLT condition is equivalent to $||T(x) - T(y)|| \leq C||x - y||$ for all x, y. This is otherwise known as the *Lipschitz continuity*, which means that there is an upper bound to the factor by which T can expand distances. This is a strong form of continuity. So, you should think of the BLT condition as a stronger form of continuity.

If $\cancel{||T||}$ C < 1, then a Lipschitz continuous mapping would be called a *contraction* mapping.

Exercise: In finite dimensions, any linear transformation is continuous, equivalently, a BLT. This is not true in infinite dimensions.

$\bar{x} \mapsto A\bar{x}$

For non-linear transformations, bounded, in Banach norm sense, and continuity have little to do with one another. It is the linearity that brings boundedness and continuity together here.

$\|T\| = C \iff \forall x \ \|Tx\| \leq C\|x\|$

$\forall \epsilon > 0 \ \exists x \neq 0 \ \frac{\|Tx\|}{\|x\|} \geq C - \epsilon$

Lecture 7:

Recall:

For NVS X and Y, a linear transformation $T : X \to Y$ is *bounded (BLT)* if there exists $C > 0$ s.t. for all $x \in X$, $||Tx|| \leq C||x||$.

Defn: The *(operator) norm* of a BLT is defined

$$||T|| := \sup_{x \in X : x \neq 0} \frac{||Tx||}{||x||}$$

Sometimes we write $||T|| := ||T||_O$.

Proposition:

$$||T|| = \sup_{x \in X : ||x|| = 1} ||Tx|| = \inf\{C > 0 : ||Tx|| \leq C||x|| \ \forall x \in X\}$$

Proof: The first equality follows from linearity of T and homogeneity of $|| \cdot ||$: write

$$\frac{||Tx||}{||x||} = ||T(\frac{x}{||x||})||$$

For the second equality, let C_{inf} be the RHS.

For all $x \neq 0$ and all $C > 0$ "as above" $\frac{||Tx||}{||x||} \leq C$ and so $\frac{||Tx||}{||x||} \leq C_{\text{inf}}$. So,

$$||T|| = \sup_{x \in X : x \neq 0} \frac{||Tx||}{||x||} \leq C_{inf}.$$

By definition of $||T||$, for all x, $||Tx|| \leq ||T|| \ |x||$. Thus, $||T||$ is one of the $C > 0$ "as above" and so

$$C_{inf} \leq ||T||. \square$$

Examples of BLT:

1. $T : \ell_\infty \to \ell_\infty$

$$T(x_1, x_2, \ldots,) = (a_1x_1, a_2x_2, \ldots,)$$

is linear. Then, T is a BLT iff a_n is bdd. And $||T|| = \sup |a_n|$.

Proof: If a_n is bdd., then for all x,

$$||Tx|| = \sup |a_n x_n| \leq (\sup |a_n|)(\sup |x_n|) = (\sup |a_n|)||x||$$

and so $||T|| \leq \sup |a_n|$.

Let $e_n = (0, \ldots, 0, 1, 0, \ldots)$ where the 1 is in the n-th coordinate. Then

$$||Te_n|| = |a_n|$$

and since $||e_n|| = 1$, $||T|| \geq \sup_n ||Te_n|| = \sup |a_n|$.

So, $||T|| = \sup |a_n|$.

If a_n is unbdd., then

$$||T|| \geq \sup_{n \in \mathbb{N}} ||Te_n|| = \sup |a_n| = \infty.$$ So, T is not a BLT.

$\square$

In particular, $T(x) = (x_1, 2x_2, 3x_3, \ldots)$ is linear but not bounded.

Can you see directly why T is not continuous?

2. Exercise (modification of Example 1): For $g \in L^\infty$, define $T : L^\infty \to L^\infty$ by $T(f) = fg$. Then T is a BLT and $||T|| = ||g||_\infty$.

bounded

Defn: A BLT $T : X \to K$ is called a ~~b~~*linear functional (BLF)*

3. Define $T : (C([0,1], || \cdot ||_{\text{sup}}) \to \mathbb{R}$ by

$$Tf = f(0).$$

Then,

$$||Tf|| = |f(0)| \leq \sup |f| = ||f||$$

So, $||T|| \leq 1$.

But for any constant function $f(x) = c$, then

$$||Tf|| = |c| = \sup |f| = ||f||$$

and so $||T|| \geq 1$.

4. Deine $T : (C([0,1], ||\cdot||_{\text{sup}}) \to \mathbb{R}$ by

$$Tf = \int_0^1 f d\mu$$

where μ is Lebesgue measure. Then,

$$||Tf|| = |\int_0^1 f d\mu| \leq \int_0^1 |f| d\mu \leq \sup |f| = ||f||$$

So, $||T|| \leq 1$.

But for any constant function $f(x) = c$, then

$$||Tf|| = |c| = \sup |f| = ||f||$$

and so $||T|| \geq 1$.

Examples 3 and 4 are special cases of the following: Let μ be a finite signed Borel measure on $[0,1]$. Define

$$Tf = \int_0^1 f d\mu$$

Then T is a BLT with $||T|| = |\mu|$, the total variation of μ.

Example 3: $\mu = \delta_0$

Example 4: $\mu =$ Lebesgue measure

It turns out that any BLT $T : (C([0,1], ||\cdot||_{\text{sup}}) \to \mathbb{R}$ is of this form, i.e.,

$$T(f) = \int f d\mu$$

for some finite signed measure μ. This is the *Riesz Representation Theorem* 7.17 in Folland.

Write $L^p((X, \mu); \mathbb{R})$ or $L^p((X, \mu); \mathbb{C})$ according to whether you are thinking of L^p as consisting of real or complex valued functions. In the next example we write $L^p = L^p((X, \mu); \mathbb{R})$.

5. Let $1 < p < \infty$, q satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Here, q is called the *dual exponent* of p and satisfies

$$q - 1 = \frac{q}{p}, \; q = \frac{p}{p-1}, 1 < q < \infty$$

For $g \in L^q$, define $T : L^p \to \mathbb{R}$ by

$$T(f) = \int fg d\mu$$

We will show that

$$||T|| = ||g||_q.$$

By Holder,

$$||T(f)|| = |T(f)| = |\int fg d\mu| \leq \int |fg| d\mu \leq ||f||_p \; ||g||_q.$$

Thus, $||T|| \leq ||g||_q$. (in particular, fg is integrable.

Let $f = |g|^{q/p}\mathrm{sgn}(g)$. We Claim $f \in L^p$ and

$$T(f) = ||f||_p \; ||g||_q$$

and so $||T|| = ||g||_q$.

Lecture 8:

Review:

Let μ be countaing measure on $\mathbb{N}$.

$$\ell^{\infty}_{\cancel{\infty}} = L^\infty(\mathbb{N}, \mu),$$

the set of bounded sequences $(x_1, x_2, \ldots)$.

$$\ell^{p}_{\cancel{p}} = L^p(\mathbb{N}, \mu),$$

the set of squences $(x_1, x_2, \ldots)$ s.t.

$$\sum_{n=1}^{\infty} |x_n|^p < \infty$$

Recall: we wanted to show that for $T : L^p \to \mathbb{R}$, $g \in L^q$, $T(f) := \int fg d\mu$, then

$$||T|| = ||g||_q.$$

Recall that we already showed $||T|| \leq ||g||_q$. So, T is ~~BED~~ BLT

Let $f = |g|^{q/p}\text{sgn}(g)$. We Claim $f \in L^p$ and

$$T(f) = ||f||_p \; ||g||_q$$

and so $||T|| = ||g||_q$.

Proof of Claim:

$$|f|^p = (|g|^{q/p}|\text{sgn}(g)|)^p = |g|^q;$$

Since $g \in L^q$,

$$||f||_p = (\int |f|^p d\mu)^{1/p} = (\int |g|^q d\mu)^{1/p} = (||g||_q)^{q/p} \qquad (1)$$

and in particular $f \in L^p$. Also, since $q - 1 = \frac{q}{p}$,

$$fg = |g|^{q/p} g \ \mathrm{sgn}(g) = |g|^{q/p+1} \neq |g|^q$$

and so

$$T(f) = \int fg d\mu = \int |g|^q d\mu = ||g||_q^q$$

$$= ||g||_q \left(||g||_q\right)^{q-1} = ||g||_q \left(||g||_q\right)^{q/p} = ||g||_q \ ||f||_p,$$

the latter equality by (1). $\square$

L(X,Y):

Defn: For NVS X, Y, $L(X, Y)$ denotes the set of all BLT from X to Y.

Propoxsition: $L(X, Y)$ is an NVS with the operator norm $|| \cdot ||_O$

Proof:

a. $L(X, Y)$ is a vector space: if $T, S \in L(X, Y)$ and $a, b \in K$, then $aT + bS$ is linear and for all $x \in X$,

$$||(aT + bS)(x)|| = ||aT(x) + bS(x)|| \leq |a| \ ||T(x)|| + |b| \ ||S(x)||$$

$$\leq (|a| \ ||T|| + |b| \ ||S||)||x||$$

Thus, $aT + bS$ is a BLT with norm $||aT + bS|| \leq |a| \ ||T|| + |b| \ ||S||$.

b. The operator norm is a norm:

i. Positivity: By definition, clearly $|| \cdot || \geq 0$ and $||0|| = 0$. If $||T|||| = 0$, then for all x, $Tx = 0$ and so $T = 0$.

ii. Homogeneity:

$$||\lambda T|| = \sup_{x \in X: x \neq 0} \frac{||\lambda T x||}{||x||} = |\lambda| \sup_{x \in X: x \neq 0} \frac{|Tx||}{||x||} = |\lambda| \ ||T||$$

iii. Triangle inequality: already proven above with $a = b = 1$. $\square$

Convergence in the operator norm on $L(X,Y)$., $T_n \to T$: means that $||T_n - T|| \to 0$. This is equivalent to $\sup_{||x||=1} ||T_n(x) - T(x)|| \to 0$, which means uniform convergence of T_n to T just over the unit sphere, equivalently the closed unit ball.

Theorem: If Y is a Banach space, so is $L(X,Y)$.

So, this gives more examples of Banach spaces. Most important example is: $L(X,K)$ is a Banach space.

Proof of Theorem: Very similar to proof that $B(\Omega)$ or L^∞ is a Banach space.

Let $T_n \in L(X,Y)$ be Cauchy. For $x \in X$,

$$||T_n(x) - T_m(x)|| \leq ||T_n - T_m||\ ||x||$$

and so $\{T_n(x)\} \in Y$ is Cauchy and so converges to some point. Define $T : X \to Y$ by $T(x) = \lim_n T_n(x)$.

It suffices to prove the
Claim: a. $T \in L(X,Y)$ and b. $T_n \to T$ in $L(X,Y)$.

Deja vu?

a. First observe that T is linear:

$$T(ax + a'x') = \lim_n T_n(ax + a'x') = \lim_n aT_n(x) + a'T_n(x')$$

$$= a \lim_n T_n(x) + a' \lim_n T_n(x') = aT(x) + a'T(x')$$

Next, since T_n is Cauchy, $||T_n||$ converges (recall that this holds in any NVS). Since $||\cdot||$ is continuous, for all $x \in X$,

$$||T(x)|| = ||\lim_n T_n(x)|| = \lim_n ||T_n(x)|| \leq (\lim_n ||T_n||)||x||$$

Thus, T is bounded with $||T|| \leq \lim_n ||T_n||$. So $T \in L(X,Y)$.

(In fact, one can show that $||T|| = \lim_n ||T_n||$).

b. For all $x \in S_1$, since $||\cdot||$ is continuous,

$$||T_n(x) - T(x)|| = ||T_n(x) - \lim_{m\to\infty} T_m(x)||$$

$$= \lim_{m\to\infty} ||T_n(x) - T_m(x)|| \leq \limsup_{m\to\infty} ||T_n - T_m||$$

So,

$$||T_n - T|| \leq \limsup_{m\to\infty} ||T_n - T_m||$$

Thus, since T_n is Cauchy,

$$\lim_{n\to\infty} ||T_n - T|| \leq \lim_{n\to\infty} \limsup_{m\to\infty} ||T_n - T_m|| = 0 \quad \square$$

Hamel Bases

Defn: A *Hamel basis* for a vector space X is a subset B such that every $x \in X$ can be expressed uniquely as a (finite) linear combination of elements of B.

This is equivalent to saying that B is i) *spanning* (i.e., every element of X can be expressed as a (finite) linear combination of X) and ii) *linearly independent* (i.e., no non-trivial linear combination of elements of B can equal 0).

So, a Hamel basis is the same as an ordinary basis as typically defined in linear algebra classes. But a Hamel basis can be infinite or even uncountable.

Theorem: Every vector space X has a Hamel basis. More generally, if Y is a subspace of X any Hamel basis for Y can be extended to a Hamel basis for X.

The proof depends on Zorn's Lemma: If S is a partially ordered set and every totally ordered subset of S has an upper bound in S, then S must have a maximal element.

Terminology of orderings $\leq$ on a set S:

– *partial ordering:* i) If $s \leq t$ and $t \leq r$, then $s \leq r$; ii) if $s \leq t$ and $t \leq s$, then $s = t$, and iii) $s \leq s$.

– *total ordering:* partial ordering + every pair of elements is comparable, i.e., given s and t, either $s \leq t$ or $t \leq s$.

– $s < t$ means $s \leq t$ and $s \neq t$.

—— Note that any finite totally ordered set of n distinct elements must satisfy $s_1 < s_2 < \ldots < s_n$.

– *upper bound* for a totally ordered subset $C \subseteq S$ is an element $u \in S$ s.t. $c \leq u$ for all $u \in C$

– *maximal element* $m \in S$ means: if $m \leq s$, then $m = s$.

Zorn's lemma is equivalent to the axiom of choice and thus is independent of the Z-F axioms of set theory.

The typical application of Zorn's lemma is to a collection of subsets of another set ordered by inclusion $A \subseteq B$. , a partially ordered set.