Lecture 33:

Riesz Representation Theorm for $C(\Omega)$: Let Ω be a compact metric space. Then $N(\Omega)$ is the space of all finite signed Borel measures on Ω. Then, $C(\Omega)^*$, with the operator norm, is isometrically isomorphic to $N(\Omega)$, with the total variation norm, is

The isometric isomorphism is given by $N(\Omega) \to C(\Omega)^*,\ \mu \mapsto \Phi_\mu$ where

$$\Phi_\mu(f) = \int f d\mu$$

and μ is a finite signed Borel measure.

Approximation Lemma: Let B be a Borel subset of a metric space Ω and μ a finite positive measure on Ω. Given $\epsilon > 0$, there exists $f \in C(\Omega)$ s.t. $0 \le f \le 1$ and $\int |f - 1_B| d\mu < \epsilon$.

Proof:

Since a positive Borel measure measure is regular, there is a compact set C, open set U s.t. $C \subset B \subset U$ and $\mu(U \setminus C) < \epsilon$.

Define

$$f = f_{C,U}(x) = \left\{ \begin{array}{cc} 0 & x \notin U \\ \frac{d(x,U^c)}{d(x,U^c)+d(x,C)} & x \in U \end{array} \right\}$$

Then f is continuous, $0 \le f \le 1, f|_C = 1, f|_{U^c} = 0$. Then

$$\int |f - 1_B| d\mu \le \mu(U \setminus C) < \epsilon. \square$$

Outline of Proof of Riesz Representation Theorem:

Step 1: Show that $\Phi_\mu(f) := \int f d\mu$ is a BLF

Clearly Φ_μ is linear. And

$$|\Phi_\mu(f)| = |\int f d\mu^+ - \int f d\mu^-| \le \int |f| d\mu^+ + \int |f| d\mu^-$$

$$\leq ||f||_{\sup}(\mu^+(X) + \mu^-(X)) = ||f||_{\sup}||\mu||$$

So, Φ_μ is a BLF with $||\Phi_\mu|| \leq ||\mu||$.

Step 2: Show that $||\Phi_\mu|| = ||\mu||$.

Let P and N be the pos. and neg. sets of the Hahn decomposition of μ.

By approximation lemma, find continuous $0 \leq f, g \leq 1$ where s.t.

$$\int |1_P - f| d|\mu| < \epsilon, \int |1_N - g| d|\mu| < \epsilon$$

Then

$$|\Phi_\mu(f-g)| = |\int (f - 1_P) d\mu - \int (g - 1_N) d\mu + \int (1_P - 1_N) d\mu|$$

$$\geq -\int |f - 1_P| d|\mu| - \int |g - 1_N| d|\mu| + |\int (1_P - 1_N) d\mu| > -2\epsilon + ||\mu||$$

Thus, Φ_μ is a BLF with $||\Phi_\mu|| = ||\mu||$. $\square$

Step 3: Show that the map $\mu \mapsto \Phi_\mu$ is linear.

$$\Phi_{a\mu+b\nu}(f) = \int f d(a\mu + b\nu) = a\int f d\mu + b\int g d\nu = a\Phi_\mu(f) + b\Phi_\nu(f)$$

Step 4: Show that the map $\mu \mapsto \Phi_\mu$ is injective.

Follows from Steps 2 and 3.

Alternatively, if $\Phi_\mu = \Phi_\nu$, then for all continuous f, $\int f d\mu = \int f d\nu$; apply approximation lemma to show that for all Borel sets B, $\mu(B) = \nu(B)$.

Step 5: Show that the map $\mu \mapsto \Phi_\mu$ is surjective.

Defn: A BLF Φ on $C(X)$ is *positive* if for all $f \in C(X)$ whenever $f \geq 0$, $\Phi(f) \geq 0$.

Step 5a. Given a BLF Φ show that you can find positive BLFs $\Phi^{\pm}$ s.t. $\Phi = \Phi^{+} - \Phi^{-}$. This is a kind of Jordan decomposition.

Step 5b (the hard part). Show that given a positive BLF Φ, there exists a positive finite measure μ s.t. for all $f \in C(X)$, $\Phi(f) = \int f d\mu$. Apply this to Φ^{+}, Φ^{-}.

Step 5bi:

Defn: For $f \in C(X)$ and U open, write $f < U$ if $0 \leq f \leq 1$ and $\overline{\{x : f(x) \neq 0\}} \subset U$.

Let Φ be a positive element of $C(X)^{*}$. For an open U define

$$\mu(U) = \sup\{\Phi(f) : f \in C(X), f < U\}$$

Step 5bii: For $E \subseteq X$, define

$$\mu^{*}(U) = \inf\{\mu(U) : U \supseteq E, U \text{ open }\}$$

Show that μ^{*} is an outer measure.

Step 5biii: Show that every open set U is μ^{*}-measurable and $\mu^{*}(U) = \mu(U)$.

Step 5biv: Define μ to be the measure on Caratheodory measurable sets.

Step 5bv: Show that indeed, for all $f \in C(X)$,

$$\Phi(f) = \int f d\mu.$$

Lecture 34:

Recall:

Riesz Representation Theorem:

Ω: a compact metric space.

$N(\Omega)$: the space of all finite signed Borel measures on Ω, with the total variation norm.

$C(\Omega)^*$, with the operator norm.

Then, $C(\Omega)^*$ is isometrically isomorphic to $N(\Omega)$.

Approximation Lemma: Let F be a Borel subset of a metric space Ω and μ a finite positive measure on Ω. Given $\epsilon > 0$, there exists $f \in C(\Omega)$ s.t. $0 \leq f \leq 1$ and $\int |f - 1_F| d\mu < \epsilon$.

Defn: Let $M(\Omega)$ denote the set of all (positive) Borel probability measures on Ω.

For $\mu \in M(\Omega)$, $||\mu|| = 1$ and so $M(\Omega) \subset B^*$, the unit ball in $C(\Omega)^*$.

Theorem: $M(\Omega)$ is a weak*-compact convex subset of B^*.

Proof: Convexity is obvious.

By Banach-Alaoglu, B^* is weak*-compact. So, it is enough to show that $M(\Omega)$ is weak* closed.

Let $\mu \in \overline{M(\Omega)}^{wk*}$. We want to show that $\mu \in M(\Omega)$, i.e., $\mu(\Omega) = 1$ and μ is a positive measure.

Suppose that μ is not a positive measure and so it has a Hahn decomposition s.t. $\mu^-(N) > 0$. Let $0 \leq f \leq 1$ be as in the approximation lemma s.t. $\int |f - 1_N| d|\mu| < \epsilon$, and so

$$|\int f d\mu + \int 1_N d\mu^-| = |\int (f - 1_N) d\mu|$$

$$\leq \int |f - 1_N| d|\mu| < \epsilon$$

and so

$$\int f d\mu < -\mu^-(N) + \epsilon$$

The set

$$U = U_{f,\mu,\epsilon} = \{\nu \in C(\Omega)^* : |\int f d\nu - \int f d\mu| < \epsilon\}$$

is a weak* neighbourhood of μ. So there exists some $\nu \in U_{f,\mu,\epsilon} \cap M(\Omega)$.

Since $\int f d\nu \geq 0$, we have

$$-\mu^-(N) + \epsilon > \int f d\mu > f d\nu - \epsilon > -\epsilon$$

Thus,

$$-\mu^-(N) > -2\epsilon$$

a contradiction, for sufficiently small ϵ.

So, μ is a positive measure.

Also, since $U_{1,\mu,\epsilon}$ is a weak* nbhd of μ, there exists $\nu \in U_{1,\mu,\epsilon} \cap M(\Omega)$. Thus,

$$|1 - \mu(X)| = |\nu(X) - \mu(X)| = |\int 1 d\nu - \int 1 d\mu| < \epsilon$$

Thus, $\mu(X) = 1$ and so μ is a probabilty measure and so $M(\Omega)$ is weak* closed. $\square$

Theorem: Let Ω be compact metric. Then $M(\Omega)$ is weak* metrizable with an explicit metric: Let $f_1, f_2, \ldots$ be a countable dense set in $C(\Omega)$.

$$d(\mu, \nu) := \sum_{n=1}^{\infty} \frac{|\int f_n d\mu - \int f_n d\nu|}{2^n ||f_n||_{\sup}}$$

Rough idea of proof: Weak* topology is smallest topology s.t. for each $f \in C(\Omega)$, $\mu \mapsto \int f d\mu$ is continuous. So, a nbhd base for weak* topology at μ is the collection of all $\cap_{i=1}^{m} U_{f_i,\mu,\epsilon_i}$ where

$$U = U_{f_i,\mu,\epsilon_i} = \{\nu \in M(\Omega) : |\int f_i d\nu - \int f_i d\mu| < \epsilon_i\}$$

For ν to be in U for m large means that the m-th partial sum of $d(\mu, \nu)$ is "small" and the tail of the series is automatically "small;" so $d(\mu, \nu)$ is "small."

Conversely for $d(\mu, \nu)$ to be "small" means that ν belongs to U for large m.

More details on the proof in Lecture 37.

Corollary: $M(\Omega)$ is weak* sequentially compact, i.e. every sequence in $M(\Omega)$ has a weak* limit point in $M(\Omega)$.

Defn: A *measure-preserving transformation (MPT)* on a *probability* space $(W, \mathcal{A}, \mu)$ is map $T : W \to W$ which is measurable and measure-preserving, i.e., for $A \in \mathcal{A}$, $T^{-1}(A) \in \mathcal{A}$ and $\mu(T^{-1}(A)) = \mu(A)$.

Ergodic theory is the study of iterations of MPTs.

Example 1: Doubling map (w.r.t. normalized Lebesgue measure on the unit interval)

$W = [0, 1)$, $\mathcal{A}$ = Borel σ-algebra, μ = Lebesgue measure
$T(x) = 2x \bmod 1$
Draw graph, which has two pieces of slope 2:

The map is MPT because inverse image of an interval I is the union of two intervals each with length $(1/2)\ell(I)$.

Chaotic map; sensitive to initial conditions.

Can also be viewed as a continuous map, an expansion, from circle to itself ($z \mapsto z^2$).

Example 2: Rotation:

$M : [0, 2\pi)$ with normalized Lebesgue measure

$T_\alpha(\theta) = \theta + \alpha \bmod 2\pi$, $\alpha \in M$.

Graph has slope 1.

The map is MPT because Lebesgue measure is translation invariant.

Can also be viewed as a continuous map, a rotation, from circle to itself ($z \mapsto e^{i\alpha} z$)

Not chaotic because it is a rigid motion.

If $\alpha/\pi \in \mathbb{Q}$, then T is periodic; each orbit is a finite set of points. If $\alpha/\pi \notin \mathbb{Q}$, then each orbit is dense.