Lecture 30:

Recall:

Lemma: Let W be a (norm-) closed subspace of an NVS X. Let $y \in X \setminus W$. Then there is an $f \in X^*$ s.t. $f(y) > 0$ and $f|_W = 0$.

Theorem; A subspace of an NVS is weakly closed iff it is norm-closed.

Proof: Every weakly closed set is norm closed.

For the converse, let W be a norm-closed subsapce of X. We must show that $\overline{W}^{wk} = W$. The weak closure can only be larger than the norm closure.

Suppose $y \in \overline{W}^{wk} \setminus W$. By the Lemma, there exists $f \in X^*$ s.t. $f(y) > 0, f|_W = 0$.

Then $\{x \in X : f(x) \neq 0\}$ is a weakly open set which contains y but is disjoint from W. Thus by Prop 2, $y \notin \overline{W}^{wk}$, a contradiction. $\square$

Theorem above can be generalized from subspaces to convex sets, using separation theorems which are consequences of HB Theorem.

Weak convergence:

Recall that $x_n \to x$ in a given topology $(X, \mathcal{T})$, means that for all $U \in \mathcal{T}$, there exists N s.t. for all $n \geq N$ $x_n \in U$.

Prop: For a NVS, $x_n \to x$ in the weak topology, written $x_n \xrightarrow{w} x$, iff for all $f \in X^*$, $f(x_n) \to f(x)$.

Proof:

"only if:" Suppose that $x_n \to x$ in the weak topology.

Let $f \in X^*$. Then for all $\epsilon > 0$, there exists N s.t. for all $n \geq N$, $x_n \in U = U_{f,x,\epsilon} = \{y \in X : |f(y) - f(x)| < \epsilon\}$. Thus, $f(x_n) \to f(x)$.

"if:" Suffices to show that for all U in a nbhd. base at x, there exists N s.t. for all $n \geq N$ $x_n \in U$.

So, suffices to show that $f_1, \ldots, f_m \in X^*$ and $\epsilon_1 > 0, \ldots, \epsilon_m > 0$, there exists N s.t. for all $n \geq N$, $x_n \in \cap_{i=1}^m U_{f_i,x,\epsilon_i}$.

But this is true: by assumption, given $\epsilon_1 > 0, \ldots, \epsilon_m > 0$, choose N_i s.t. for all $n \geq N_i$, each $|f_i(x_n) - f_i(x)| < \epsilon_i$ and so for $n \geq \max(N_1, \ldots, N_m)$, $x_n \in \cap_{i=1}^m U_{f_i,x,\epsilon_i}$. $\square$

Norm convergence, written $x_n \xrightarrow{n} x$, means $||x_n - x|| \to 0$.

Note: $\mathcal{T}^{norm}$ is stronger than $\mathcal{T}^{weak}$. So, norm convergence is stronger than weak convergence.

Sanity check: if $x_n \xrightarrow{n} x$ in the norm topology and $f \in X^*$, then

$$|f(x_n) = f(x)| \leq ||f|| \, ||x_n - x|| \to 0.$$

Example: a weakly convergent sequence that is not norm convergent.

Recall $(\ell^2)^* = \ell^2$; for $x \in \ell^2, x^* \in (\ell^2)^*$

$$x^*(x) = \langle x, x^* \rangle = \sum_{i=1}^{\infty} x_i x_i^*$$

Let e_n be the standard basis in sequence spaces.

$$e_1 = (1, 0, 0, \ldots,), e_2 = (0, 1, 0, \ldots,)$$

For each $x^* \in (\ell^2)^*$,

$$x^*(e_n) = x_n^* \to 0$$

So, $e_n \xrightarrow{w} 0$.

But for $n \neq m$, $||e_n - e_m||_2 = \sqrt{2}$ and so e_n does not converge in the norm topology.

Theorem: For an NVS X, $(X, \mathcal{T}^{weak})$ is a locally convex (i.e., has a base of convex sets), Haudorff TVS.

Proof:

1. TVS: Continuity of addition: if $x_n \xrightarrow{w} x$ and $y_n \xrightarrow{w} y$, then for all $f \in X^*$,

$$f(x_n + y_n) = f(x_n) + f(y_n) \to f(x) + f(y) = f(x + y)$$

and so $x_n + y_n \xrightarrow{w} x + y$.

Continuity of scalar multiplication: if $x_n \xrightarrow{w} x$ and $\lambda_n \to \lambda$, then then for all $f \in X^*$

$$f(\lambda_n x_n) = \lambda_n f(x_n) \to \lambda f(x) = f(\lambda x)$$

and so $\lambda_n x_n \xrightarrow{w} \lambda x$.

WARNING: We have only proven sequential continuity of vector addition and scalar multiplication. Since the weak topology need not be metric, this may not suffice. Can give a legitimate proof of continuity of addition and scalar multiplication, using Nets (Section 4.3). A net $x_i, i \in I$ is a "generalized sequence" where I is a generalized index set, called a directed set. More on this next time.

2. Hausdorff: By Hahn-Banach given any $x \neq y$, there is a $f \in X^*$ s.t. $f(x - y) \neq 0$. Then $x \in U_{f,x,(1/2)|f(x-y)|}$, $y \in U_{f,y,(1/2)|f(x-y)|}$, and $U_{f,x,(1/2)|f(x-y)|} \cap U_{f,y,(1/2)|f(x-y)|} = \emptyset$

3. Locally convex: We first claim that each $U_{f,x_0,\epsilon}$ is convex: suppose $x, y \in U_{f,x_0,\epsilon}$. Let $t \in [0, 1]$. Then,

$$|f(tx + (1-t)y - x_0)| = |(tf(x - x_0) + (1-t)f(y - x_0)||$$

$$\leq t|f(x - x_0)| + (1-t)|f(y - x_0)| < t\epsilon + (1-t)\epsilon = \epsilon;$$

so, $tx + (1-t)y \in U_{f,x_0,\epsilon}$ and so $U_{f,x_0,\epsilon}$ is convex.

Since intersections of convex sets are convex and sets of the form $\cap_{i=1}^{n} U_{f_i,x_i,\epsilon_i}$ form a base, we have a base of convex sets. $\square$

Theorem: For an NVS X, $(X, \mathcal{T}^{norm})$ is a locally convex, Haudorff TVS.

Proof: we already proved TVS. Hausdorf follows from fact that norm topology is metric. Locally convex proven similarly to locally convex for weak topology using norm instead of linear functionals.

Lecture 31:

Recall:

$$\overline{B}_1(0) = \bigcap_{\{f \in X^*: ||f||=1\}} \{x : |f(x)| \leq 1\}$$

This implies

$$\overline{B}_M(0) = \bigcap_{\{f \in X^*: ||f||=1\}} \{x : |f(x)| \leq M\}$$

and so if a subset $A \subset X$ is bounded with $M = \sup_{x \in A} \ ||x|| < \infty$, then

$$\overline{A}^{wk} \subseteq \overline{B_M(0)}$$

Defn: Given a NVS X, the *weak* topology* on X^* is the weakest topology such that each $\hat{x}, x \in X$ is continuous on X^*.

Remarks:

1. The weak topology on X^* is is the weakest topology such that each $f \in X^{**}$ is continuous on X^*.

2. The weak* topology on X^* is is the weakest topology such that each $\hat{x} \in \hat{X} := \{\hat{x} : x \in X\}$ is continuous on X^*. Note $\hat{X} \subseteq X^{**}$.

3. The weak* topology is weaker than the weak topology which is weaker than the norm topology.

4. The weak* topology lives only on NVSs which are dual spaces and therefore only on Banach spaces.

5. It lives on all reflexive Banach spaces because they are all dual spaces.

Theorem: For an NVS X, $(X, \mathcal{T}^{norm})$, $(X, \mathcal{T}^{wk}$, and $(X^*, \mathcal{T}^{weak*})$ is a locally convex, Haudorff TVS.

We proved this for the norm and weak topologies last time. The proof for weak* is similar to proof for weak. But all three follow from a more general result in Folland:

Defn: Let $\{p_\alpha\}_{\alpha\in A}$ be a collection of semi-norms on a vector space X. The *semi-norm topology* $\mathcal{T}$ is the topology with base consisting of all finite intersections of sets of the form:

$$U_{\alpha,x_0,\epsilon} := \{x \in X : \ p_\alpha(x - x_0) < \epsilon\}$$

Theorem 5.14: The semi-norm topology is a locally convex TVS.

Prop 5,16a: The semi-norm topology is Hausdorff iff for each $x \in X$, $x \neq 0$, there exists $\alpha \in A$ s.t. $p_\alpha(x) \neq 0$, then .

Examples of semi-norm topologies: Given an NVS X,

– Norm topology: $\{p_\alpha\} = \{|| \cdot ||\}$

– Weak topology: $\{p_\alpha\} = \{|f| : f \in X^*|\}$

– Weak* topology: $\{p_\alpha\} = \{|\hat{x}| : x \in X\}$

$|| \cdot ||$ and $|f|$ are clearly semi-norms, and
$|\hat{x}|(f) := |f(x)|$ is a semi-norm on X^*.

Proof:

$$|\hat{x}|(f+g) = |f(x) + g(x)| \leq |f(x)| + |g(x)| = |\hat{x}|(f) + |\hat{x}|(g)$$

$$|\hat{x}|(\lambda f) = |\lambda| \ |f(x)| = |\lambda| \ |\hat{x}|(f) \quad \square$$

To check the Hausdorff condition for each topology above:
– Norm topology: If $x \neq 0$, then $||x|| \neq 0$.

– Weak topology: If $x \neq 0$, by Hahn–Banach Corollary, there exists $f \in X^*$ s.t. $f(x) = ||x|| \neq 0$.

– Weak* topology: If $f \in X^*$ and $f \neq 0$, then there exists $x \neq 0$ s.t. $\hat{x}(f) = f(x) \neq 0$.

Proof of Prop 5.16a: If $x \neq y$, $p_\alpha(x - y) \neq 0$, then $U_{\alpha,x,(1/2)p_\alpha(x-y)}$ and $U_{\alpha,y,(1/2)p_\alpha(x-y)}$ are disjoint.

The proof of Theorem 5.14 follows along the lines of how we proved the result for the weak topology. That relied on the fact that $x_n \xrightarrow{w} x$ iff for all $f \in X^*$ $f(x_n) \to f(x)$.

In fact, we needed the corresponding result for nets $x_i, i \in I$.

For a general semi-norm topology, the proof of Theorem 5.14 is based on:

FACT: the net $x_i \to x$ iff for all $\alpha \in A$ $p_\alpha(x - x_i) \to 0$.

Let's say a bit more about nets.

Defn: A *directed set* is a set with an ordering that is reflexive and transitive and s.t. every pair of elements has an upper bound.

Main examples of directed sets:
– $\mathbb{N}$ with usual ordering
– $\mathbb{N} \times \mathbb{N}$ with ordering $(x, y) \leq (x', y')$ iff $x \leq x'$ and $y \leq y'$.

Defn: A *net* in a topological space $(X, \mathcal{T})$ is a map from a directed set I to X: $i \mapsto x_i$

Defn: A net $(x_i)_{i \in I}$ converges to $x \in X$, denoted $x_i \to x$, if for all nbhds $U \ni x$, there exists $i \in I$ s.t. for all $j \geq i$ (in the ordering that defines the directed set), $x_j \in U$.

Recall: Characterization of continuity (Prop 4.19). $f : (X, \mathcal{T}) \to (Y, \mathcal{S})$ if for every net $x_i \to x$ in X, the net $f(x_i) \to f(x)$.

Note: In metric spaces, sequences can replace nets in the characterization of continuity.

Lecture 32:

Given a NVS X, the weak sequential closure of a set A, denoted $\overline{A}^{wks}$ is the collection of limit points of sequences in A.

Fact: $\overline{A}^{wks} \subset \overline{A}^{wk}$ because given a convergent sequence $x_n \xrightarrow{w} x$, with $x_n \in A$, then every nbhd U of x contains some x_n and thus intersects A.

Example: a set $A \subset X$ s.t. $\overline{A}^{wk} \setminus \overline{A}^{wks} \neq \emptyset$.

$X = \ell^2$. Let e_n be the standard basis vectors, let

$$A = \{e_m + me_n : 1 \leq m < n, n \geq 2\}$$

	6	$e_1 + e_6$	$e_2 + 2e_6$	$e_3 + 3e_6$	$e_4 + 4e_6$	$e_5 + 5e_6$
	5	$e_1 + e_5$	$e_2 + 2e_5$	$e_3 + 3e_5$	$e_4 + 4e_5$	
	4	$e_1 + e_4$	$e_2 + 2e_4$	$e_3 + 3e_4$		
	3	$e_1 + e_3$	$e_2 + 2e_3$			
n	2	$e_1 + e_2$				
		1	2	3	4	5
		m				

Recall that $(\ell^2)^* = \ell^2$.

Claim: No sequence in A converges weakly to 0, so $0 \notin \overline{A}^{wks}$.

Proof: Recall $x_k \xrightarrow{w} 0$ means that for all $y \in \ell^2$,

$$\langle x_k, y \rangle \to 0.$$

Let $e_{m_k} + m_k e_{n_k}$, $m_k < n_k$ be any sequence in A.

Case 1: m_k bounded

Let M be an upper bound. Let $y = e_1 + \ldots e_M$. Then

$$\langle e_{m_k} + m_k e_{n_k}, y \rangle = \langle e_{m_k} + m_k e_{n_k}, e_1 + \ldots e_M \rangle =$$

$$1 + m_k \langle e_{n_k}, e_1 + \dots e_M \rangle \geq 1$$

and so $e_{m_k} + m_k e_{n_k} \overset{w}{\not\to} 0$.

Case 2: m_k unbounded

Then n_k unbounded.

Choose a subsequence s.t. m_{k_i} and n_{k_i} are strictly increasing with i

Let $y = \sum_i \frac{1}{m_{k_i}} e_{n_{k_i}} \in \ell^2$ because $m_{k_i} \geq i$ and so

$$\sum_i \frac{1}{m_{k_i}^2} \leq \sum_i \frac{1}{i^2} < \infty$$

Thus

$$\langle e_{m_{k_i}} + m_{k_i} e_{n_{k_i}}, y \rangle \geq m_{k_i} \frac{1}{m_{k_i}} = 1$$

and so $e_{m_{k_i}} + m_{k_i} e_{n_{k_i}} \overset{w}{\not\to} 0$.

Claim: 0 is in the weak closure of A. So, $0 \in \overline{A}^{wk} \setminus \overline{A}^{wks}$

Proof:

Recall nbhd. basis at 0: $\cap_{i=1}^k U_{y^i,0,\epsilon_i}$ where $y^1, \dots, y^k \in \ell^2$ and

$$U_{y^i,0,\epsilon_i} = \{x \in \ell^2 : |\langle x, y^i \rangle| < \epsilon_i\}$$

Now,

$$\langle e_m + m e_n, y^i \rangle = y^i_m + m y^i_n, \; i = 1, \dots, k$$

Given $\epsilon_i > 0$, choose m s.t. $|y^i_m| < \epsilon_i/2$ i =1, …, k and then choose n s.t. $m|y^i_n| < \epsilon_i/2$, i =1, …, k . Then

$$|\langle e_m + m e_n, y^i \rangle| < \epsilon_i, \; i = 1, \dots, k$$

So, $e_m + m e_n \in A \cap (\cap_{i=1}^k U_{y^i,0,\epsilon_i})$

Thus for every nbhd. U of 0, there exist m, n s.t. $e_m + me_n \in A \cap U$.

Thus, 0 is in the weak closure of A. $\square$

Banach-Alaoglu Theorem: For any NVS X, the unit ball in X^* is weak* compact.

Proof relies on Tychanoff's Theorem: The product of compact spaces is compact.

Proof of Banach-Alaoglu Theorem:

Let $P = \times_{x \in X}[-||x||, ||x||]$, the product space with product topology coming from the Euclidean topology on each interval $[-||x||, ||x||]$. By Tychanoff's theorem, P is compact.

Let B^* denote the unit ball in X^*:

$$B^* = \{f \in X^* : ||f|| \leq 1\}$$

Here, $||f||$ is of course the operator norm.

Let

$$\tau : B^* \to P, \ \tau(f) = (f(x))_{x \in X}$$

Note that $\tau(f)$ does indeed belong to P since $||f|| \leq 1$ and so for all $x \in X$, $|f(x)| \leq ||f|| \, ||x|| \leq ||x||$.

Also, τ is an injection since $\{f(x) : x \in X\}$ completely determine f, and so τ is a bijection onto its image in P.

So, B^* may be regarded as a subset, namely, $\tau(B^*)$ of P.

Moreover, the weak* topology of B^* agrees with the relative topology from the product topology on P: continuity of the projections in P on $\tau(B^*)$ corresponds exactly to continuity of the maps $\hat{x}, x \in X$ on B^*.

Thus, since any closed subset of a compact space is compact, to show that B^* is weak* compact it suffices to show that $\tau(B^*)$ is closed in the product topology.

Observe that $\tau(B^*)$ is exactly the set of all elements of P that can be regarded as linear functionals, i.e., $y \in \tau(B^*)$ iff

$$(*) \quad \text{given } x_1, x_2, x_3 \in X \text{ and } \alpha_1, \alpha_2 \in K \ s.t.$$

$$x_3 = \alpha_1 x_1 + \alpha_2 x_2, \text{ we have } y_{x_3} = \alpha_1 y_{x_1} + \alpha_2 y_{x_2}$$

So let $y \in \overline{\tau(B^*)}^{wk*}$. We show that (*) holds as follows.

Given any $\epsilon > 0$,

$$U := \{z \in P : |z_{x_3} - y_{x_3}| < \epsilon, |z_{x_2} - y_{x_2}| < \epsilon, |z_{x_1} - y_{x_1}| < \epsilon\}$$

is a nbhd of y in P and thus intersects $\tau(B^*)$. Let $z \in U \cap \tau(B^*)$. Then, $z_{x_3} = \alpha_1 z_{x_1} + \alpha_2 z_{x_2}$ and so

$$|y_{x_3} - \alpha_1 y_{x_1} - \alpha_2 y_{x_2}| \leq |y_{x_3} - z_{x_3} - \alpha_1(y_{x_1} - z_{x_1}) - \alpha_2(y_{x_2} - z_{x_2})|$$

$$\leq |y_{x_3} - z_{x_3}| + |\alpha_1|\ |y_{x_1} - z_{x_1}| + |\alpha_2|\ |y_{x_2} - z_{x_2}| < \epsilon(1 + |\alpha_1| + |\alpha_2|)$$

Letting $\epsilon \to 0$, we see that

$$y_{x_3} = \alpha_1 y_{x_1} + \alpha_2 y_{x_2}$$

Thus, $y \in \tau(B^*)$ and so $\tau(B^*)$ is closed and thus compact. $\square$

Note: you can simplify the last part of this argument by using nets.