Lecture 3:
At end of last lecture, we were in the course of proving that \((B(\Omega), ||\cdot||_{sup})\) is a Banach space.
We proved:
\[ a. \ B(\Omega) \text{ is a vector space, as a subspace of } K^\Omega. \]

Now we prove:
\[ b. \ ||f|| = ||f||_{sup} \text{ is a norm and } c. \text{ The norm metric is complete.} \]

Proof of b:
- Positivity: clearly \( ||f|| \geq 0 \) and \( ||f|| = 0 \) iff \( f = 0 \).
- Homogeneity: \( ||\lambda f|| = |\lambda||f|| \),
- Subadditivity:

\[
||f + g|| = \sup\{|f(x) + g(x)| : x \in \Omega\} \leq \sup\{|f(x)| + |g(x)| : x \in \Omega\} \\
\leq \sup(||f|| + ||g||) = ||f|| + ||g||
\]

Intuitively, \( ||f + g|| \) may be less than \( ||f|| + ||g|| \) because of cancellations.

c. The norm metric is complete.

Time out for a general NVS fact:

Lemma: For any Cauchy sequence \( \{x_n\} \) in a NVS, the sequence \( ||x_n|| \) converges in \( \mathbb{R} \) and in particular is bounded.
Proof: By reverse triangle inequality,

\[
|| ||x_n|| - ||x_m|| || \leq ||x_n - x_m||
\]

and so \( \{||x_n||\} \) is Cauchy in \( \mathbb{R} \) and thus converges in \( \mathbb{R} \). \( \Box \)

Note: convergence in \( B(\Omega) \) is the topology of uniform convergence: \( ||f_n - f|| \to 0 \) iff \( f_n \) converges uniformly to \( f \). We sometimes denote the norm by \( ||\cdot||_u \) instead of \( ||\cdot||_{sup} \).
Proof of completeness of $B(\Omega)$: Let $f_n$ be Cauchy in $B(\Omega)$. In particular, for each fixed $x \in \Omega$, the sequence $f_n(x)$ is Cauchy in $\mathbb{R}$ because:

$$|f_n(x) - f_m(x)| \leq \sup_y |f_n(y) - f_m(y)| = ||f_n - f_m||$$

Since $\mathbb{R}$ is complete, $f_n(x)$ converges to some $a_x \in \mathbb{R}$. Define $f(x) = a_x$.

Will show that 1) $f \in B(\Omega)$ and 2) $||f - f_n|| \to 0$.

Proof of 1:

$$|f(x)| = \lim_n |f_n(x)| \leq \sup_n |f_n(x)| \leq \sup_n ||f_n|| < \infty$$

by Lemma above. So, $f \in B(X)$.

Proof of 2 (uniform Cauchy + pointwise convergence implies uniform convergence)

Let $\epsilon > 0$. Let $N$ be s.t. for all $m, n \geq N$, $||f_n - f_m|| < \epsilon$. For fixed $x \in \Omega$, choose $m = m_x \geq N$ s.t. $|f_m(x) - f(x)| < \epsilon$. Then

$$|f_n(x) - f(x)| \leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)|$$

$$\leq ||f_n - f_m|| + |f_m(x) - f(x)| < 2\epsilon.$$

and so

$$||f_n - f|| = \sup_x |f_n(x) - f(x)| \leq 2\epsilon.$$

So, $||f - f_n|| \to 0.$ □

Alternative proof of 2 (without $\epsilon$'s):

$$|f_n(x) - f(x)| = \lim_{m \to \infty} |f_n(x) - f_m(x)| \leq \limsup_{m \to \infty} ||f_n - f_m||$$

Thus,

$$||f_n - f|| \leq \limsup_{m \to \infty} ||f_n - f_m||$$

13
Thus,
\[
\lim_{n \to \infty} \|f_n - f\| \leq \lim_{n \to \infty} \limsup_{m \to \infty} \|f_n - f_m\| = 0. \quad \square
\]

Time out for another general Banach space fact:

Proposition: A subspace of a Banach space is a Banach space iff it is closed.

Proof: Use:
- A subspace $Y$ of a Banach space $X$ inherits the NVS of $X$.
- A subset of a complete metric space is complete iff it is closed.

3. Let $\Omega$ be a metric space.

\[ BC(\Omega) = \{\text{bounded continuous } f : \Omega \to K\} \]

is a Banach space with sup norm.

Proof:

a. $BC(\Omega)$ is a subspace of $B(\Omega)$: check closure of vector addition and scalar multiplication.

b. $BC(\Omega)$ is closed in $B(\Omega)$: A uniform limit of cts. functions is cts. And a uniform limit of bounded functions is bounded. So $BC(\Omega)$ is a closed and therefore a complete subspace of $B(\Omega)$. \quad \square

4. Let $\Omega$ be a compact metric space. Continuous functions, $C(\Omega)$, on $\Omega$, forms a Banach space.

Proof: $C(\Omega) = BC(\Omega)$ since $\Omega$ is compact.

5. Let $\Omega$ be a metric space. Continuous functions that vanish at infinity:

\[ C_0(\Omega) = \{f : X \to R : \text{cts, } \forall \varepsilon > 0 \exists \text{ compact } L \subset \Omega : \forall x \notin L |f(x)| < \varepsilon\} \]

is a Banach space with sup norm.
Proof:

a. check $C_0(\Omega)$ is a subspace of $BC(\Omega)$:
check closure of vector addition and scalar multiplication.

- vector addition: given $f, g \in C_0(\Omega)$, $f + g$ is cts. Given $\epsilon > 0$, let $L_{f,\epsilon}, L_{g,\epsilon}$ be the corresponding compact sets. Then $L := L_{f,\epsilon/2} \cup L_{g,\epsilon/2}$ is compact and for $x \notin L$, both $|f(x)|, |g(x)| < \epsilon$ and so $|f(x) + g(x)| < \epsilon$

- scalar multiplication: $L_{\lambda f,\epsilon} = L_{f,\epsilon/|\lambda|}$.

b. check $C_0(\Omega)$ is a closed subset of $BC(\Omega)$:

Let $f_n \in C_0(\Omega)$ s.t. $f_n$ converges uniformly to $f$. Then $f$ is cts.
Let $\epsilon > 0$. Then for some $N$, $||f_N - f|| < \epsilon$.

Then there is a compact set $L_{f_N,\epsilon}$ for $f_N$ and $\epsilon$,

i.e., $|f_N(x)| < \epsilon \ \forall x \notin L_{f_N,\epsilon}$.

Then for $x \notin L_{f_N,\epsilon}$,

$$|f(x)| \leq |f(x) - f_N(x)| + |f_N(x)| \leq ||f - f_N|| + |f_N(x)| < 2\epsilon$$

So, $f \in C_0(\Omega)$, and so $C_0(\Omega)$ is a closed subspace of $BC(\Omega)$ and thus is a Banach space. □

6. Continuous functions with compact support:

Defn: The **support** of a function $f$ is $supp(f) = \{x \in X : f(x) \neq 0\}$.

$$C_c(\Omega) = \{f : \Omega \rightarrow R : \text{cts s.t. supp}(f) \text{ is compact}\}$$

$C_c(\Omega) \subset C_0(\Omega)$ because for any $f \in C_c(\Omega)$ and $x \notin supp(f)$, $f(x) = 0$.

And $C_c(\Omega)$ is a subspace of $C_0(\Omega)$ because $supp(f + g) \subset supp(f) \cup supp(g)$ and thus is closed subset of the union of two compact sets, and thus is compact.
Thus $C_c(\Omega)$ inherits NVS from $C_0(\Omega)$.

But if $\Omega$ is not compact, then $C_c(\Omega)$ need not be a Banach space because it need not be closed:

Example: $\Omega = \mathbb{R}$;

$$f_n(x) = \begin{cases} 
  e^{-x^2} & x \in [-n, n] \\
  \text{straight line from } (n, e^{-n^2}) \text{ to } (n + 1, 0) \\
  \text{straight line from } (-n, e^{-n^2}) \text{ to } (-n - 1, 0) \\
  0 & x \not\in [-n - 1, n + 1]
\end{cases}$$

Then $f_n \in C_c(\mathbb{R})$ converges uniformly to $e^{-x^2}$ which does not have compact support. □

In fact, $C_c(\mathbb{R})$ is dense in $C_0(\mathbb{R})$.

In a finite-dimensional space all subspaces are closed and the only subspace. But, as we have seen, in an infinite-dimensional NVS, there can be proper dense subspaces (in particular, not closed).
Lecture 4:

Recall Banach spaces: $B(\Omega, \| \cdot \|_{\text{sup}})$ and its subspaces

with $\Omega$ metric: $BC(\Omega)$, $C(\Omega)$ (with $\Omega$ compact), $C_0(\Omega)$ and the NVS $C_c(\Omega)$.

Note: Proof of completeness of $B(\Omega)$, showing that $\|f_n - f\| \to 0$ (uniform convergence), can be simplified (no $\epsilon$’s needed).

One way or another, the idea is to show “uniform Cauchy + point-wise convergence implies uniform convergence.”

7. Let $\Omega = \mathbb{N}$ with discrete metric. All subsets are open.
   a. $B(\mathbb{N}) = \ell_\infty$ is the set of bounded sequences $(x_1, x_2, \ldots)$.
   b. $BC(\mathbb{N}) = B(\mathbb{N}) = \ell_\infty$ because for the discrete metric, all subsets are open and so all functions on $\mathbb{N}$ are continuous.
   c. $C_0(\mathbb{N}) = c_0$ is the set of sequences that converge to 0.

   Proof: For any $S \subset \mathbb{N}$, $\{\{s\} : x \in S\}$ is an open cover for of $S$ that has no proper open subcover. Thus, $S$ is compact iff it is finite.

   If $x = (a_1, a_2, \ldots) \in c_0$, then for all $\epsilon > 0$, there is a finite set $S \subset \mathbb{N}$ s.t. for $s \in \mathbb{N} \setminus S$, $|x_s| \leq \epsilon$ and thus for all $s \geq |S|$, $|x_s| \leq \epsilon$ Thus $a_n \to 0$.

   Conversely, if $a_n \to 0$, then for all $\epsilon > 0$, there exists $N$ s.t. for all $s > N$, $|x_s| < \epsilon$ and so $x \in c_0$.

   d. $C_c(\mathbb{N})) = c_c$ is the set of sequences that are eventually 0, because the compact sets are the finite sets.

   Direct proof that $c_c$ is not Banach: let $a_n = 1/n$ and $x = (a_1, a_2, \ldots) \in c_0$. Let $x^n = (a_1, \ldots, a_n, 0, \ldots)$. Then each $x^n \in c_c$ and $x^n \to x$. So, $c_c$ is not closed and in fact is a proper dense subspace of $c_0$.  

17
8. $C^1([0, 1])$: the set of all real-valued ctsly differentiable functions on $[0, 1]$ (including one-sided limits at the endpoints).

$C^1([0, 1])$ is a subspace of $C([0, 1])$ and thus inherits the sup-norm as an NVS $(C^1([0, 1]), \| \cdot \|_{\text{sup}})$.

But it is not complete because it is not closed: the uniform limit of $C^1$ functions need not be $C^1$.

Example: approximate $f(x) = |x|$ uniformly by $C^1$ functions.

In fact, $C^1([0, 1])$ is a dense subspace of $C([0, 1])$ (use Stone-Weierstrass).

However, \[ \| f \| := \| f \|_{\text{sup}} + \| f' \|_{\text{sup}} \] is a norm that makes $C^1([0, 1])$ complete.

Proof: If $f_n$ is Cauchy in $(C^1([0, 1]), \| \cdot \|)$, then both $f_n$ and $f'_n$ are uniformly Cauchy and thus $f_n \to f$, $f'_n \to g$ uniformly for some $f, g \in C([0, 1])$. Then from Math 321, $g = f'$ and so $\| f_n - f \| \to 0$. □

9. Let $D$ be the open unit disk and $\overline{D}$ the closed unit disk in $\mathbb{C}$.

\[ X = \{ f : \overline{D} \to \overline{D} : f \text{ analytic on } D \text{ and extends ctsly to } \overline{D} \} \]

is a Banach space, with the sup norm.

- Proof: $X$ is a subspace of $C(\overline{D})$ because a linear combination of analytic functions is analytic and is a closed subspace because a uniform limit of analytic functions is analytic. □

Time out for a defn and a theorem:
Defn: Let $X$ be a NVS and $(x_n) \in X$. The series $\sum_{n=1}^{\infty} x_n$ converges if the sequence of partial sums converges to some $x \in X$. The series converges absolutely if $\sum_{n=1}^{\infty} ||x_n|| < \infty$.

Note that on $\mathbb{R}$, abs. convergence implies convergence and this is consistent with fact that $\mathbb{R}$ is complete. Of course, even on $\mathbb{R}$, convergence does not imply abs. convergence.

Theorem (Theorem 5.1): A NVS is complete iff every absolutely convergent series is convergent.

Proof: “if.”

Let $x_n$ be a Cauchy sequence. For each $j$, choose $N_j$ such that for all $m, n \geq N_j$, $||x_m - x_n|| \leq 2^{-j}$. We may assume that the sequence $N_j$ is strictly increasing.

Let $y_j = x_{N_j+1} - x_{N_j}$. Then $y_j$ is absolutely convergent because the series $\sum_{j=1}^{\infty} ||y_j||$ is nonnegative with bounded partial sums

$$\sum_{j=1}^{K} ||y_j|| \leq \sum_{j=1}^{K} 2^{-j} \leq 1,$$

So $x_{N_{K+1}} - x_{N_1} = \sum_{j=1}^{K} y_j \to y$ for some $y$. This means that $\lim_{j \to \infty} x_{N_{j+1}} = y + x_{N_1}$, and so $x_n$ has a convergent subsequence. But any Cauchy sequence with a convergent subsequence is convergent.

“only if.”
Let $x_n$ be absolutely convergent. Thus, for all $\epsilon > 0$, there exists $N$ such that $\sum_{j=N}^{\infty} ||x_n|| < \epsilon$. Let $y_n = \sum_{j=1}^{n} x_j$. Then $y_n$ is Cauchy because for $m \geq n \geq N$,

$$||y_m - y_n|| = ||\sum_{j=n+1}^{m} x_j|| \leq \sum_{j=n+1}^{m} ||x_j|| \leq \sum_{j=N}^{\infty} ||x_j|| < \epsilon$$

Since the space is complete, $y_n$ converges and this mean that $\sum_{j=1}^{N} x_j$ is convergent.
Lecture 5:

10. Defn: For \( p \geq 1 \), \( L^p(\Omega, \mu) \)-spaces:

\[
L^p(\Omega, \mu) = \{ f : \Omega \to K : \int_\Omega |f|^p d\mu < \infty \}
\]

for \( \sigma \)-finite measure space \((\Omega, \mu)\).

Recall that \( f \in L^p \) is really an equivalence class mod \( \mu - a.e. \).

Equivalence classes mod 0. Stephen G. showed that:

\[
||f||_p := (\int_\Omega |f|^p d\mu)^{1/p}
\]

is well defined on equivalence classes and that

\( L^p \) is a Banach space:

a. \( L^p \) is a well-defined vector space, in particular closed under addition and scalar multiplication; proven by Stephen G.

b. \( L^p \) is a NVS.

c. \( L^p \) is complete.

Proof:

i. Positivity: Clearly, \( ||f||_p \geq 0 \) and \( ||0||_p = 0 \). If \( ||f||_p = 0 \), then \( |f|^p = 0 \) a.e. and so \( |f| = 0 \) a.e.

ii.

\[
||\lambda f||_p = (\int |\lambda f|^p d\mu)^{1/p} = |\lambda| (\int |f|^p d\mu)^{1/p} = |\lambda|||f||_p
\]

iii. Minkowski inequality: Stephehen G. showed \( ||f+g||_p \leq ||f||_p + ||g||_p \).

So, \( L^p \) is a NVS. □

c. \( L^p \) is complete and therefore a Banach space.
Proof: (Theorem 6.6 of Folland. Uses Theorem 5.1).
Suppose that \( f_k \in L^p \) is absolutely convergent, i.e., \( \sum_1^\infty \|f_k\|_p = B < \infty \).
We want to show that there exists some \( F \in L^p \) s.t.
\[
\|F - \sum_1^n f_k\|_p \to 0
\]
Guess: \( F = \sum_1^\infty f_k \). Must show:
1. \( F \in L^p \) and in particular \( F \) is finite a.e.
2.
\[
\|F - \sum_1^n f_k\|_p \to 0
\]
Let
\[
G_n = \sum_1^n |f_k|, \quad G = \sum_1^\infty |f_k| = \lim_n G_n
\]
which is a (possibly extended) real-valued measurable function.
By Minkowski,
\[
\|G_n\|_p \leq \sum_1^n \|f_k\|_p \leq B
\]
and so for all \( n \),
\[
\int G_n^p \leq \int f_k^p \leq B
\]
Since \( G_n^p \uparrow G^p \), by MCT,
\[
\int G^p = \lim_n \int G_n^p \leq B
\]
Thus, \( G \in L^p \) and in particular, \( G \) is finite a.e.
Since \( G \) is finite a.e., the series \( F := \sum_1^\infty f_k(x) \) converges absolutely and thus \( F \) converges a.e. and so is finite a.e.
Since $|F| \leq G$, $F \in L^p$. This gives 1 above.

For 2 above:

$$|F - \sum_{1}^{n} f_k|^p \leq (|F| + \sum_{1}^{n} |f_k|)^p \leq (2G)^p \in L^1$$

Since $\sum_{1}^{n} f_k \to F$ a.e., $|F - \sum_{1}^{n} f_k|^p \to 0$ a.e. By DCT

$$\|F - \sum_{1}^{n} f_k\|_p = (\int |F - \sum_{1}^{n} f_k|^p)^{1/p} \to 0$$

□

Examples: $\ell^p = \ell^p(N)$ are special examples of $L^p$ spaces, with counting measure on $\mathbb{N}$:

$$\ell^p = \{(a_1, a_2, \ldots) : \sum_{n} |a_n|^p < \infty\}$$

10. $L^\infty$ is a Banach space. (with essential sup norm, $\|f\|_\infty$, which is well-defined on equiv. classes of functions).

$$\|f\|_\infty = \inf\{a \geq 0 : \mu(|f|^{-1}(a, \infty)) = 0\}$$

$$L^\infty(\Omega, \mu) = \{f : \|f\|_\infty < \infty\}.$$  

Facts about $L^\infty$ (Stephen G.):

1. $\|f\|_\infty$ is well-defined on equivalence classes in $L^\infty(\Omega, \mu)$ because $\|f\|_\infty$ is insensitive to changes on sets of measure zero.

2. The inf is achieved, i.e., $\mu(|f|^{-1}(\|f\|_\infty, \infty)) = 0$.

3. $|f| \leq \|f\|_\infty \mu - a.e.$ and $\|f\|_\infty \leq \sup |f|$
4. For $f \in L^\infty$, there exists $g = f$ $\mu$-a.e. s.t.

$$||f||_\infty = \sup |g|$$

namely, $g := f \chi_{f^{-1}(\infty, ||f||\infty)}$

5. $L^\infty$ is a vector space (check that addition and scalar multiplication are well-defined, i.e., are insensitive to changes on sets of measure zero).

6. $(L^\infty, || \cdot ||_\infty)$ is an NVS: Positivity and Homogeneity are fairly obvious and Stephen proved Minkowski for $L^\infty$:

$$||f + g||_\infty \leq ||f||_\infty + ||g||_\infty$$

Exercise: show that $L^\infty(\Omega, \mu)$ is complete, and hence Banach, by modifying our proof that $B(\Omega)$ is complete.

Note: Here, we have shown completeness of $L^\infty$ by modifying the proof of completeness of $\ell_\infty$. But completeness of $\ell_\infty$ is a corollary of completeness of $L^\infty$ because $\ell_\infty = L^\infty(\mathbb{N}, \mu)$ where $\mu$ is counting measure on $\mathbb{N}$. So, we could have started by proving $L^\infty(\Omega, \mu)$ is Banach and then gotten completeness of $\ell_\infty$ and other sequence spaces.

Recall that any two norms on a finite-diml. space are equivalent.

Example of inequivalent norms on the same vector space: $(C^1, || \cdot ||_{\sup}), (C^1, || \cdot ||_{C^1})$ because one is not complete and the other is complete.
Non-compactness of unit ball:

For a subset $Y$ of a metric space $X$ and $x \in X$

$$d(x, Y) := \inf_{y \in Y} d(x, y)$$

Proposition $d(x, Y) = 0$ iff $x \in \overline{Y}$.

Proof: $d(x, Y) = 0 \iff$ there exists $y_n \in Y$ s.t. $y_n \to x \iff x \in \overline{Y}$.

□

Corollary: If $Y$ is closed and $x \not\in Y$, then $d(x, Y) > 0$.

Q: Let $X$ be an NVS, $Y$ a closed subspace of $X$. What is

$$d_{\max} := \sup_{x \in X: ||x||=1} d(x, Y)?$$

Proposition: $d_{\max} = 1$.

We will prove this next time and will use it to prove that the unit ball in any infinite-dimensional NVS $X$ is not compact.