Lecture 3:

At end of last lecture, we were in the course of proving that $(B(\Omega), ||\cdot||_{sup})$ is a Banach space.

We proved:

a. $B(\Omega)$ is a vector space, as a subspace of K^{Ω}. $K = \mathbb{R}$ or $\mathbb{C}$

Now we prove:

b. $||f|| = ||f||_{sup}$ is a norm and c. The norm metric is complete.

Proof of b:

– Positivity: clearly $||f|| \geq 0$ and $||f|| = 0$ iff $f = 0$.

– Homegeneity: $||\lambda f|| = |\lambda| ||f||$,

– Subaddivitity:

$$||f+g|| = \sup\{|f(x)+g(x)| : x \in \Omega\} \leq \sup\{|f(x)|+|g(x)| : x \in \Omega\}$$
$$\leq \sup(||f|| + ||g||) = ||f|| + ||g||$$

Intuitively, $||f+g||$ may be less than $||f|| + ||g||$ because of cancellations.

c. The norm metric is complete.

Time out for a general NVS fact:

Lemma: For any Cauchy sequence $\{x_n\}$ in a NVS, the sequence $||x_n||$ converges in $\mathbb{R}$ and in particular is bounded.

Proof: By reverse triangle inequality,

$$|\ ||x_n|| - ||x_m||\ | \leq ||x_n - x_m||$$

and so $\{||x_n||\}$ is Cauchy in $\mathbb{R}$ and thus converges in $\mathbb{R}$. $\square$

Note: convergence in $B(\Omega)$ is the topology of uniform convergence: $||f_n - f|| \to 0$ iff f_n converges uniformly to f. We sometimes denote the norm by $||\cdot||_u$ instead of $||\cdot||_{\text{sup}}$.

Proof of completeness of $B(\Omega)$: Let f_n be Cauchy in $B(\Omega)$. In particular, for each fixed $x \in \Omega$, the sequence $f_n(x)$ is Cauchy in $\mathbb{R}$ because:

$$|f_n(x) - f_m(x)| \leq \sup_y |f_n(y) - f_m(y)| = ||f_n - f_m||$$

Since $\mathbb{R}$ is complete, $f_n(x)$ converges to some $a_x \in \mathbb{R}$. Define $f(x) = a_x$.

Will show that 1) $f \in B(\Omega)$ and 2) $||f - f_n|| \to 0$.

Proof of 1:

$$|f(x)| = \lim_n |f_n(x)| \leq \sup_n |f_n(x)| \leq \sup_n ||f_n|| < \infty$$

by Lemma above. So, $f \in B(X)$.

Proof of 2 (uniform Cauchy + pointwise convergence implies uniform convergence)

Let $\epsilon > 0$. Let N be s.t. for all $m, n \geq N$, $||f_n - f_m|| < \epsilon$. For fixed $x \in \Omega$, choose $m = m_x \geq N$ s.t. $|f_m(x) - f(x)| < \epsilon$. Then

$$\begin{aligned} |f_n(x) - f(x)| &\leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)| \\ &\leq ||f_n - f_m|| + |f_m(x) - f(x)| < 2\epsilon. \end{aligned}$$

and so

$$||f_n - f|| = \sup_x |f_n(x) - f(x)| \leq 2\epsilon.$$

So, $||f - f_n|| \to 0$. $\square$

Alternative proof of 2 (without ϵ's):

$$|f_n(x) - f(x)| = \lim_{m\to\infty} |f_n(x) - f_m(x)| \leq \limsup_{m\to\infty} ||f_n - f_m||$$

Thus,

$$||f_n - f|| \leq \limsup_{m\to\infty} ||f_n - f_m||$$

Thus,

$$\lim_{n\to\infty} ||f_n - f|| \le \lim_{n\to\infty} \limsup_{m\to\infty} ||f_n - f_m|| = 0. \square$$

Time out for another general Banach space fact:

Proposition: A subspace of a Banach space is a Banach space iff it is closed.

Proof: Use:

– A subspace Y of a Banach space X inherits the NVS of X.

– A subset of a complete metric space is complete iff it is closed.

$\square$

3. Let Ω be a metric space.

$$BC(\Omega) = \{\textit{bounded continuous } f : \Omega \to K\}$$

is a Banach space with sup norm.

Proof:

a. $BC(\Omega)$ is a subspace of $B(\Omega)$: check closure of vector addition and scalar multiplication.

b. $BC(\Omega)$ is closed in $B(\Omega)$: A uniform limit of cts. functions is cts. And a uniform limit of bounded functions is bounded. So $BC(\Omega)$ is a closed and therefore a complete subspace of $B(\Omega)$. $\square$

4. Let Ω be a compact metric space. Continuous functions, $C(\Omega)$, on Ω, forms a Banach space.

Proof: $C(\Omega) = BC(\Omega)$ since Ω is compact.

5. Let Ω be a metric space. Continuous functions that vanish at infinity:

$$C_0(\Omega) = \{f : X \to R : \; cts, \; \forall \epsilon > 0 \, \exists \, \text{compact } L \subset \Omega : \; \forall x \notin L \, |f(x)| < \epsilon\}$$

is a Banach space with sup norm.

Notes: Constants $\notin C_0(\Omega)$ if Ω is not compact

Proof:

a. check $C_0(\Omega)$ is a subspace of $BC(\Omega)$:

check closure of vector addition and scalar multiplication.

– vector addition: given $f, g \in C_0(\Omega)$, $f+g$ is cts. Given $\epsilon > 0$, let $L_{f,\epsilon}, L_{g,\epsilon}$ be the corresponding compact sets. Then $L := L_{f,\epsilon/2} \cup L_{g,\epsilon/2}$ is conpact and for $x \notin L$, both $|f(x)|, |g(x)| < \epsilon$ and so $|f(x) + g(x)| < \epsilon$

$\cancel{L_{f+g,\epsilon} = cl(f+g \neq 0)}$

$\cancel{\subseteq cl((f \neq 0) \cup g \neq 0)}$

$\cancel{\subseteq cl(f \neq 0) \cup cl(g \neq 0)}$

– scalar multiplication: $L_{\lambda f, \epsilon} = L_{f, \epsilon/|\lambda|}$.

b. check $C_0(\Omega)$ is a closed subset of $BC(\Omega)$:

Let $f_n \in C_0(\Omega)$ s.t. f_n converges uniformly to f. Then f is cts. $\in BC(\Omega)$

Let $\epsilon > 0$. Then for some N, $||f_N - f|| < \epsilon$.

Then there is a compact set $L_{f_N,\epsilon}$ for f_N and ϵ, i.e., $|f_N(x)| < \epsilon \; \forall x \notin L_{f_N,\epsilon}$.

Then for $x \notin L_{f_N,\epsilon}$,

$$|f(x)| \leq |f(x) - f_N(x)| + |f_N(x)| \leq ||f - f_N|| + |f_N(x)| < 2\epsilon$$

So, $f \in C_0(\Omega)$, and so $C_0(\Omega)$ is a closed subspace of $BC(\Omega)$ and thus is a Banach space. □

6. Continuous functions with compact support:

Defn: The *support* of a function f is $supp(f) = \overline{\{x \in X : f(x) \neq 0\}}$.

$$C_c(\Omega) = \{f : \Omega \to R : \text{ cts } s.t. \; supp(f) \; is \; compact.\}$$

$C_c(\Omega) \subset C_0(\Omega)$ because for any $f \in C_c(\Omega)$ and $x \notin supp(f)$, $f(x) = 0$.

And $C_c(\Omega)$ is a subspace of $C_0(\Omega)$ because $supp(f+g) \subset supp(f) \cup supp(g)$ and thus is closed subset of the union of two compact sets, and thus is compact.

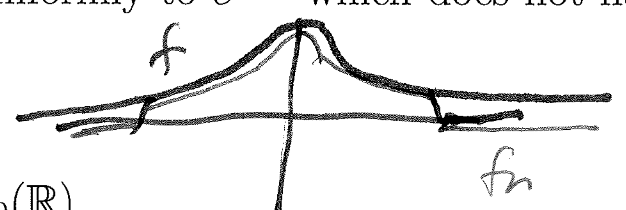
Thus $C_c(\Omega)$ inherits NVS from $C_0(\Omega)$.

But if Ω is not compact, then $C_c(\Omega)$ need not be a Banach space because it need not be closed:

Example: $\Omega = \mathbb{R}$;

$$f_n(x) = \begin{cases} e^{-x^2} & x \in [-n, n] \\ \text{straight line from } (n, e^{-n^2}) \text{ to } (n+1, 0) & \\ \text{straight line from } (-n, e^{-n^2}) \text{ to } (-n-1, 0) & \\ 0 & x \notin [-n-1, n+1] \end{cases}$$

Then $f_n \in C_c(\mathbb{R})$ converges uniformly to e^{-x^2} which does not have compact support. $\square$



In fact, $C_c(\mathbb{R})$ is dense in $C_0(\mathbb{R})$.

In a finite-dimensional space all subspaces are closed and the only subspace. But, as we have seen, in an infinite-dimensional NVS, there can be proper dense subspaces (in particular, not closed).

Lecture 4:

Recall Banach spaces: $B(\Omega, ||\cdot||_{\text{sup}})$ and its subspaces

with Ω metric: $BC(\Omega)$, $C(\Omega)$ (with Ω compact), $C_0(\Omega)$ and the NVS $C_c(\Omega)$.

Note: Proof of comleteness of $B(\Omega)$, showing that $||f_n - f|| \to 0$ (uniform convergence), can be simplified (no ϵ's needed).

One way or another, the idea is to show "uniform Cauchy + pointwise convergence implies uniform convergence."

7. Let $\Omega = \mathbb{N}$ with discrete metric. All subsets are open.

a. $B(\mathbb{N}) = \ell_\infty$ is the set of bounded sequences $(x_1, x_2, \ldots)$.

b. $BC(\mathbb{N}) = B(\mathbb{N}) = \ell_\infty$ because for the discrete metric, all subsets are open and so all functions on $\mathbb{N}$ are continuous.

c. $C_0(\mathbb{N}) = c_0$ is the set of sequences that converge to 0.

Proof: For any $S \subset \mathbb{N}$, $\{\{s\} : x \in S\}$ is an open cover for of S that has no proper open subcover. Thus, S is compact iff it is finite.

If $x = (a_1, a_2, \ldots) \in c_0$, then for all $\epsilon > 0$, there is a finite set $S \subset \mathbb{N}$ s.t. for $s \in \mathbb{N} \setminus S$, $|x_s| \leq \epsilon$ and thus for all $s \geq |S|$, $|x_s| \leq \epsilon$ Thus $a_n \to 0$.

Conversely, if $a_n \to 0$, then for all $\epsilon > 0$, there exists N s.t. for all $s > N$, $|x_s| < \epsilon$ and so $x \in c_0$.

d. $C_c(\mathbb{N})) = c_c$ is the set of sequences that are eventually 0, because the compact sets are the finite sets.

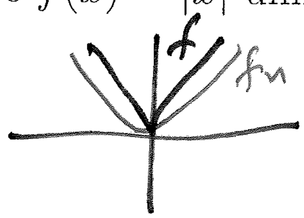
Direct proof that c_c is not Banach: let $a_n = 1/n$ and $x = (a_1, a_2, \ldots) \in c_0$. Let $x^n = (a_1, \ldots, a_n, 0, \ldots)$. Then each $x^n \in c_c$ and $x^n \to x$. So, c_c is not closed and in fact is a proper dense subspace of c_0.

8. $C^1([0,1])$: the set of all real-valued ctsly diffble. functions on $[0,1]$ (including one-sided limits at the endpoints)

$C^1([0,1])$ is a subspace of $C([0,1])$ and thus inherits the sup-norm as an NVS $(C^1([0,1]), ||\cdot||_{\text{sup}})$.

But it is not complete because it is not closed: the uniform limit of C^1 functions need not be C^1.

Example: approximate $f(x) = |x|$ uniformly by C^1 functions.



$C([-1,1])$
$C^1([-1,1])$

In fact, $C^1([0,1])$ is a dense subspace of $C([0,1])$ (use Stone-Weirstrass).

However, $||f|| := ||f||_{\text{sup}} + ||f'||_{\text{sup}}$ is a norm that makes $C^1([0,1])$ complete

Proof: If f_n is Cauchy in $(C^1([0,1]), ||\cdot||$, then both f_n and f_n' are uniformly Cauchy and thus $f_n \to f$, $f_n' \to g$ uniformly for some $f, g \in C([0,1[)$. Then from Math 321, $g = f'$ and so $||f_n - f|| \to 0$. $\square$

9. Let D be the open unit disk and $\overline{D}$ the closed unit disk in $\mathbb{C}$.

$$X = \{f : \overline{D} \to \overline{D} : f \text{ analytic on D and extends ctsly to } \overline{D}\}$$

is a Banach space, with the sup norm.

– Proof: X is a subspace of $C(\overline{D})$ because a linear combination of analytic functions is analytic and is a closed subspace because a uniform limit of analytic functions is analytic. $\square$

Time out for a defn and a theorem:

Defn: Let X be a NVS and $(x_n) \in X$. The series $\sum_{n=1}^{\infty} x_n$ *converges* if the sequence of partial sums converges to some $x \in X$. The series *converges absolutely* if $\sum_{n=1}^{\infty} ||x_n|| < \infty$.

Note that on $\mathbb{R}$, abs. convergence implies convergence and this is consistent with fact that $\mathbb{R}$ is complete. Of course, even on $\mathbb{R}$, convergence does not imply abs. convergence.

Theorem (Theorem 5.1): A NVS is complete iff every absolutely convergent series is convergent.

Proof: "if:"

$$\begin{tikzcd} \text{Sequence} & \text{Cauchy} \arrow[d] & \text{Convergent} \\ \text{Series} & \text{Abs. Conv.} \arrow[r] & \text{Convergent} \arrow[u] \end{tikzcd}$$

Let x_n be a Cauchy sequence. For each j, choose N_j such that for all $m, n \geq N_j$, $||x_m - x_n|| \leq 2^{-j}$. We may assume that the sequence N_j is strictly increasing.

Let $y_j = x_{N_{j+1}} - x_{N_j}$. Then y_j is absolutely convergent because the series $\sum_{j=1}^{\infty} ||y_j||$ is nonnegative with bounded partial sums

$$\sum_{j=1}^{K} ||y_j|| \leq \sum_{j=1}^{K} 2^{-j} \leq 1,$$

So $x_{N_{K+1}} - x_{N_1} = \sum_{j=1}^{K} y_j \to y$ for some y. This means that $\lim_{j \to \infty} x_{N_{j+1}} = y + x_{N_1}$, and so x_n has a convergent subsequence. But any Cauchy sequence with a convergent subsequence is convergent.

"only if:"

$$\begin{tikzcd} \text{Sequence} & \text{Cauchy} \arrow[r] & \text{Convergent} \arrow[d] \\ \text{Series} & \text{Abs. Conv.} \arrow[u] & \text{Convergent} \end{tikzcd}$$

Let x_n be absolutely convergent. Thus, for all $\epsilon > 0$, there exists N such that $\sum_{j=N}^{\infty} ||x_n|| < \epsilon$. Let $y_n = \sum_{j=1}^{n} x_j$. Then y_n is Cauchy because for $m \geq n \geq N$,

$$||y_m - y_n|| = ||\sum_{j=n+1}^{m} x_j|| \leq \sum_{j=n+1}^{m} ||x_j|| \leq \sum_{j=N}^{\infty} ||x_j|| < \epsilon$$

Since the space is complete, y_n converges and this mean that $\sum_{j=1}^{N} x_j$ is convergent.

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Lecture 5:

10. Defn: For $p \geq 1$, $L^p(\Omega, \mu)$- *spaces*:

$$L^p(\Omega, \mu) = \{f : \Omega \to K : \int_\Omega |f|^p d\mu < \infty\}$$

for σ-finite measure space (Ω, μ).

Recall that $f \in L^p$ is really an equivalence class mod $\mu - a.e.$.

Equivalence classes mod 0. Stephen G. showed that:

$$||f||_p := (\int_\Omega |f|^p d\mu)^{1/p}$$

is well defined on equivalence classes and that

L^p is a Banach space:

a. L^p is a well-defined vector space, in particular closed under addition and scalar multiplication; proven by Stephen G.

b. L^p is a NVS.

c. L^p is complete.

Proof:

i. Positivity: Clearly, $||f||_p \geq 0$ and $||0||_p = 0$. If $||f||_p = 0$, then $|f|^p = 0$ a.e. and so $|f| = 0$ a.e.

ii.

$$||\lambda f||_p = (\int |\lambda f|^p d\mu)^{1/p} = |\lambda|(\int |f|^p d\mu)^{1/p} = |\lambda|||f||_p$$

iii. Minkowski inequality: Stepehen G. showed $||f+g||_p \leq ||f||_p + ||g||_p$.

So, L^p is a NVS. $\square$

c. L^p is complete and therefore a Banach space.

Proof: (Theorem 6.6 of Folland. Uses Theorem 5.1).

Suppose that $f_k \in L^p$ is absolutely convergent, i.e., $\sum_1^\infty ||f_k||_p =$ $S\ \cancel{B} < \infty$,

We want to show that there exists some $F \in L^p$ s.t.

$$||F - \sum_1^n f_k||_p \to 0$$

Guess: $F = \sum_1^\infty f_k$. Must show:

1. $F \in L^p$ and in particular F is finite a.e.
2.

$$||F - \sum_1^n f_k||_p \to 0$$

Let

$$G_n = \sum_1^n |f_k|, \; G = \sum_1^\infty |f_k| = \lim_n G_n$$

which is a (possibly extended) real-valued measurable function.

By Minkowski,

$$||G_n||_p \leq \sum_1^n ||f_k||_p \leq \cancel{B}\ S$$

and so for all n,

$$\int G_n^p \leq \cancel{B^p}\ S^p$$

Since $G_n^p \uparrow G^p$, by MCT,

$$\int G^p = \lim_n \int G_n^p \leq \cancel{B^p}\ S^p$$

Thus, $G \in L^p$ and in particular, G is finite a.e.

Since G is finite a.e., the series $F := \sum_1^\infty f_k(x)$ converges absolutely and thus F converges a.e. and so is finite a.e.

a.e.

Since $|F| \leq G$, $F \in L^p$. This gives 1 above.

For 2 above:

$$|F - \sum_1^n f_k|^p \leq (|F| + \sum_1^n |f_k|)^p \leq (2G)^p \in L^1$$

Since $\sum_1^n f_k \to F$ a.e., $|F - \sum_1^n f_k|^p \to 0$ a.e. By DCT

$$||F - \sum_1^n f_k||_p = (\int |F - \sum_1^n f_k|^p)^{1/p} \to 0$$

$\square$

Examples: $\ell^p = \ell^p(N)$ are special examples of L^p spaces, with counting measure on $\mathbb{N}$:

$$\ell^p = \{(a_1, a_2, \ldots) : \sum_n |a_n|^p < \infty\}$$

10. L^∞ is a Banach space. (with essential sup norm, $||f||_\infty$, which is well-defined on equiv. classes of functions).

$$||f||_\infty = \inf\{a \geq 0 : \mu(|f|^{-1}(a, \infty)) = 0\}$$

$$L^\infty(\Omega, \mu)) = \{f : \ ||f||_\infty < \infty).$$

Facts about L^∞(Stephen G.):

1. $||f||_\infty$ is well-defined on equivalence classes in $L^\infty(\Omega, \mu))$ because $||f||_\infty$ is insensitive to changes on sets of measure zero.

2. The inf is achieved, i.e., $\mu(|f|^{-1}(||f||_\infty, \infty)) = 0$.

3.

$$|f| \leq ||f||_\infty \mu - a.e. \text{ and } ||f||_\infty \leq \sup |f|$$

4. For $f \in L^\infty$, there exists $g = f$ μ-a.e. s.t.

$$||f||_\infty = \sup |g|$$

   namely, $g := f\chi_{f^{-1}(-\infty, ||f_\infty||])}$

5. L^∞ is a vector space (check that addition and scalar multiplication are well-defined, i.e., are insensitive to changes on sets of measure zero).

6. $(L^\infty, ||\cdot||_\infty)$ is an NVS: Positivity and Homogeneity are fairly obvious and Stephen proved Minkowski for L^∞:

$$||f + g||_\infty \leq ||f||_\infty + ||g||_\infty$$

Exercise: show that $L^\infty(\Omega, \mu)$ is complete, and hence Banach, by modifying our proof that $B(\Omega)$ is complete.

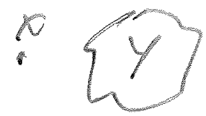
Note: Here, we have shown completeness of L^∞ by modifying the proof of completeness of ℓ_∞. But completeness of ℓ_∞ is a corollary of completeness of L^∞ because $\ell_\infty = L^\infty(\mathbb{N}, \mu)$ where μ is counting measure on $\mathbb{N}$. So, we could have started by proving $L^\infty(\Omega, \mu)$ is Banach and then gotten completeness of ℓ_∞ and other sequence spaces.

Recall that any two norms on a finite-diml. space are equivalent.

Example of inequivalent norms on the same vector space: $(C^1, ||\cdot||_{\text{sup}}), (C^1, ||\cdot||_{C^1})$ because one is not complete and the other is complete.

Non-compactness of unit ball:

For a subset Y of a metric space X and $x \in X$



$$d(x, Y) := \inf_{y \in Y} d(x, y)$$

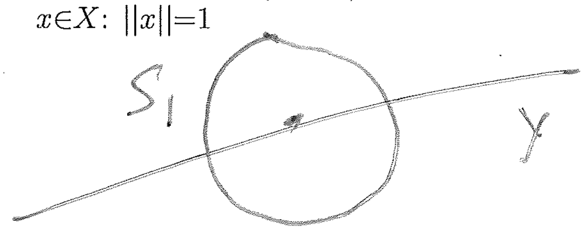
Proposition $d(x, Y) = 0$ iff $x \in \overline{Y}$.

Proof: $d(x, Y) = 0 \Leftrightarrow$ there exists $y_n \in Y$ s.t. $y_n \to x \Leftrightarrow x \in \overline{Y}$.
$\square$

Corollary: If Y is closed and $x \notin Y$, then $d(x, Y) > 0$.

Q: Let X be an NVS, Y a proper closed subspace of X. What is

$$d_{max} := \sup_{x \in X:\ ||x||=1} d(x, Y)?$$



Proposition: $d_{max} = 1$.

We will prove this next time and will use it to prove that the unit ball in any infinite-dimensional NVS X is *not* compact.