Lecture 12:
Recall:
Real Hahn-Banach Theorem: Let $W$ be a subspace of $X$, a real vector space. Let $f: W \to \mathbb{R}$ be a linear functional. Let $p: X \to \mathbb{R}$ be a sublinear functional such that for all $x \in W$, $f(x) \leq p(x)$. Then $f$ can be extended to a linear functional $F$ on $X$ s.t. $F(x) \leq p(x)$ for all $x \in X$ and $F|_W = f$.

Prop 1: Let $X$ be a complex vector space.

a. Let $f$ be a $\mathbb{R}$-linear functional. Then $f_{\mathbb{C}}(x) := f(x) - if(ix)$ is a $\mathbb{C}$-linear functional, the complexification of $f$.

b. Let $h$ be a $\mathbb{C}$-linear functional and let $f = \mathbb{R}h$. Then $f$ is a $\mathbb{R}$-linear functional and $h(x) = f_{\mathbb{C}}(x)$.

\[ \mathbb{R} \quad \mathbb{C} \]
\[ f = \mathbb{R}h \iff h = f_{\mathbb{C}} \]

Prop 2a: Let $f$ be a $\mathbb{R}$-linear functional. Let $p(x)$ be a $\mathbb{C}$-seminorm on $X$. Then

\[ \forall x \ |f(x)| \leq p(x) \iff \forall x \ |f_{\mathbb{C}}(x)| \leq p(x) \]

Complex Hahn-Banach Theorem: Let $X$ be a complex vector space and $p(x)$ a $\mathbb{C}$-semi-norm on $X$.

Let $W$ be a complex subspace of $X$.

Let $h$ be $\mathbb{C}$-linear-functional on $W$ s.t. $\forall x \ |h(w)| \leq p(w)$.

Then there is a $\mathbb{C}$-linear-functional $H$ on $X$ s.t. $H|_W = h$ and $\forall x \ |H(x)| \leq p(x)$. 

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\[
\begin{align*}
\mathbb{R} & \quad \mathbb{C} \\
X & \quad F \quad \rightarrow \quad H = F_{\mathbb{C}} \\
\uparrow & \\
W & \quad f = \Re h \quad \leftarrow \quad h
\end{align*}
\]

Proof of Complex HB:

Let \( f = \Re h \), and so by Prop 1b, \( h = f_{\mathbb{C}} \).

Then \( f \) is a \( \mathbb{R} \)-linear functional on \( W \) s.t.
\[
\forall w \in W, \quad f(w) \leq |f(w)| \leq |h(x)| \leq p(w).
\]

Since \( p \) is a \( \mathbb{C} \)-semi-norm, it is a \( \mathbb{R} \)-semi-norm and hence a semi-linear functional.

By real HB, there is a \( \mathbb{R} \)-linear functional \( F \) on \( X \) s.t.
\[
F|_W = f \quad \text{and} \quad \forall x \in X, \quad F(x) \leq p(x).
\]

But then for all \( \forall x \in X \quad |F(x)| \leq p(x) \) (when \( F(x) \) is negative, apply \( F(x) \leq p(x) \) to \( -x \)).

Let \( H := F_{\mathbb{C}} = F(x) - iF(ix) \). Then,

\[
H|_W(x) = (F(x) - iF(ix))|_W = f(x) - if(ix) = f_{\mathbb{C}}(x) = h(x).
\]

And by Prop. 2a, since \( \forall x \in X \quad |F(x)| \leq p(x) \), we have \( \forall x \in X \quad |H(x)| \leq p(x) \).

\[\square\]

Theorem ((Folland Theorem 5.8b,c)) Consequences of HB Theorem

1. Let \( x \in X \) s.t. \( x \neq 0 \). There exists a BLF \( f \) on \( X \) s.t. \( ||f|| = 1 \)
and \( f(x) = ||x|| \).

2. The BLFs separate points in \( X \), i.e., for \( x, y \in X \), there exists a BLF \( f \) on \( X \) s.t. \( f(x) \neq f(y) \).
So, there are lots of BLFs.

Proof:
1: Define $g$ on $W := Kx$ by $g(rx) = r||x||$, $r \in K$. Then for all $w \in W$, $|g(w)| = ||w||$.

Apply H-B with $p(u) = ||u||$, to get a BLF $f$ s.t. $f|_W = g$ and $|f(u)| \leq ||u||$ for all $u \in X$.

So, $||f|| \leq 1$. But $f(x) = g(x) = ||x||$, and so $||f|| = 1$.

2. Let $x, y \in X, x \neq y$. Let $f$ be a BLF as in part 1 s.t. $f(x - y) = ||x - y|| \neq 0$. Then $f(x) \neq f(y)$.

**Time out: Isomorphisms in different categories of math:**

Isomorphism is a map that is
1. Bijective (or nearly bijective)
2. Preserves a structure
3. Inverse preserves structure

Sometimes, 3 is automatic.

1. Defn: A vector space isomorphism is a bijective linear map from one vector space $X$ to another $Y$, with linear inverse.

Fact: A bijective linear map automatically has a linear inverse and thus is a vector space isomorphism.

Proof: Let $T : X \to Y$ be a bijective linear transformation.

Let $y_1, y_2 \in Y, a_1, a_2 \in K$ and $T^{-1}$ be the inverse of $T$ as a set mapping. Let

$$x = T^{-1}(a_1y_1 + a_2y_2), x_1 = T^{-1}(y_1), x_2 = T^{-1}(y_2)$$
Then

\[ T(x) = a_1y_1 + a_2y_2 = a_1T(x_1) + a_2T(x_2) = T(a_1T^{-1}(y_1) + a_2T^{-1}(y_2)) \]

the latter equality because \( T \) is linear. Since \( T \) is injective, we get

\[ T^{-1}(a_1y_1 + a_2y_2) = x = (a_1T^{-1}(y_1) + a_2T^{-1}(y_2)) \]

So, \( T^{-1} \) is linear. \( \square \)

2. Defn: A \textit{homeomorphism} is a bijective continuous map from one metric space \( X \) (more generally, topological space) to another \( Y \), with continuous inverse.

Fact: A bijective continuous map need not have a continuous inverse and thus need not be a homeomorphism.

Example:

\( T : [0, 2\pi) \to S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \), \( T(\theta) = (\cos(\theta), \sin(\theta)) \)

\( T \) is bijective because it maps onto the unit circle and \( (\cos(\theta), \sin(\theta)) \) uniquely determine \( \theta \) in \([0, 2\pi)\). \( T \) is continuous because \( \cos(\theta) \) and \( \sin(\theta) \) are continuous. But \( T^{-1} \) is not continuous, since \( T^{-1}(1, 0) = 0 \) and for \( \epsilon > 0 \), \( T^{-1}(\sqrt{1 - \epsilon^2}, -\epsilon) \) is close to \( 2\pi \).

An exception:

(Math 320) A bijective continuous map from one compact metric space to another automatically has a continuous inverse and therefore is a homeomorphism.

3. Defn: A \textit{homeomorphic isomorphism} (isomorphism) is a bijective continuous linear map from one NVS to another, with a continuous inverse.
Fact: A bijective continuous linear map need not have a continuous inverse and so need not be a homeomorphism isomorphism:

Example in HW2 (4d). A Bijective BLT with whose inverse is not a BLT.

An exception:

Corollary of Open Mapping Theorem: A bijective continuous linear map from one Banach space to another automatically has a continuous inverse and therefore is a homeomorphic isomorphism.

4. Notion of Sameness for NVS:

Defn: An isometric isomorphism is a bijective norm-preserving linear map from one NVS to another that has a norm-preserving inverse.

Fact: A bijective norm-preserving linear map automatically has a norm-preserving inverse:

You effectively need to prove this in Hw2, 6a.

Note: In HW, I defined isometric isomorphism, without the norm-preserving inverse.

Isometric isomorphism:
1. Linear
2. Norm-preserving
3. Injective
4. Surjective

Note: 2 implies 3: if \( \Psi(x) = 0 \), then \( ||x|| = ||\Psi(x)|| = ||0|| = 0 \) and so \( \Psi \) has trivial kernel.

Also, sometimes we refer to an isometric isomorphism onto its image, so we don’t need to prove 4.
Lecture 13:

Defns: The dual space of an NVS $X$ is defined

$$X^* = \mathcal{L}(X, K) = \{\text{BLFs } f : X \rightarrow K\}$$

Note that $X^*$ is a Banach space, with the operator norm topology, even if $X$ is not, because "$Y$ is complete" implies that "$L(X, Y)$ is complete."

Defn: The double dual space of an NVS $X$ is

$$X^{**} = (X^*)^*$$

For $x \in X$, define $\hat{x} \in X^{**}$ by

$$\hat{x}(f) = f(x)$$

Prop: $\hat{x} \in X^{**}$, i.e. $\hat{x}$ is a BLF on $X^*$.

Proof: $\hat{x}$ a linear functional:

$$\hat{x}(af + bg) = (af + bg)(x) = af(x) + bg(x) = a\hat{x}(f) + b\hat{x}(g).$$

Since

$$|\hat{x}(f)| = |f(x)| \leq \|f\| \|x\| = \|x\| \|f\|$$

we have

$$\|\hat{x}\| \leq \|x\|.$$

and so $\hat{x}$ is a BLF on $X^*$. □

In fact, Prop: $\|\hat{x}\| = \|x\|$.

Proof: Enough to find $f$ s.t. $|\hat{x}(f)| = \|x\| \|f\|$.

By Theorem above (Folland 5.8b), there exists a linear functional $f$ on $X$ s.t. $\|f\| = 1$ and $f(x) = \|x\|$. So,

$$|\hat{x}(f)| = |f(x)| = \|x\| = \|x\| \|f\|$$
and so $||\hat{x}|| = ||x||$. □

Note: $||\hat{x}|| = ||\hat{x}||_{X^{**}}, ||x|| = ||x||_X$.

Defn: The canonical embedding is defined:

$$X \rightarrow X^{**}, x \mapsto \hat{x}$$

Theorem (Folland Theorem 5.8d): The canonical embedding $X \rightarrow X^{**}, x \mapsto \hat{x}$ is an isometric isomorphism onto its image $Y$.

$X$ and $Y$ are the “same,”

Proof:

Linearity: for all $f \in X^*$,

$$\overline{ax + by}(f) = f(ax + by) = af(x) + bf(y) = a\hat{x}(f) + b\hat{y}(f) = (a\hat{x} + b\hat{y})(f)$$

Norm-preserving: as mentioned above, $||\hat{x}|| = ||x||$.

Injective: as mentioned above, 1-1 follows from norm-preserving. □

Proposition: A NVS is Banach iff the image of its canonical embedding is closed in $X^{**}$.

Proof: $X^{**}$ is Banach since it is the dual of $X^*$.

So, the image $Y$ of the NVS $X$ is closed iff $Y$ is complete iff $X$ is complete, the latter “iff” since $X$ and $Y$ are isometrically isomorphic and so one is complete iff the other is complete □

Note that in any event an NVS embeds as a dense subspace of the closure $\overline{Y}$ of the image $Y$ of canonical embedding, which is a Banach space. Deja vu?

Defn: An NVS is reflexive if the image of the canonical imbedding is $X^{**}$. We sometimes write this as $X = X^{**}$, having identified $X$ with its image.
Note that a reflexive NVS is automatically a Banach space because the image of its canonical embedding is $X^{**}$, which is trivially closed.

Alternatively, $X^{**}$ is Banach because it is a dual space. Reflexive Banach spaces are rare but have lots of nice properties.

Example of a dual space:

Theorem (Folland Thm 6.15) Let $1 < p < \infty$, $1/p + 1/q = 1$. Then, $(L^p)^* = L^q$. More precisely, $(L^p)^*$ and $L^q$ are isometrically isomorphic.

Corollary $(L^p)^{**} = (L^q)^* = (L^p)$, and so $L^p$ is reflexive.

Sketch of proof of: $(L^p)^* = L^q$:

Let $g \in L^q$.

$$L_g : L^p \to \mathbb{R}, L_g(f) = \int f g d\mu$$

Recall (from awhile back): $L_g$ is a BLF (and thus $L_g \in (L^p)^*$) and $\|L_g\| = \|g\|_q$.

Let $\Psi : L^q \to (L^p)^*$, $g \mapsto L_g$

Proposition: $\Psi$ is an isometric isomorphism.

So, $(L^p)^* = L^q$.

Proof of Prop:

Linearity: $g \mapsto L_g$ is linear:

$$L_{ag_1 + bg_2}(f) = \int (ag_1 + bg_2)(f) d\mu = a \int g_1 f d\mu + b \int g_2 f d\mu$$

$$= a L_{g_1}(f) + b L_{g_2}(f)$$
and so

\[ L_{a}g_{1} + b g_{2} = a L_{g_{1}} + b L_{g_{2}} \]

Norm-preserving: \( ||L_g|| = ||g||_q \).

Injective: as noted above, injective follows from norm-preserving.

Surjective: The hard part (Theorem 6.15) Rough idea:
Say \( L^p = L^p(\Omega, \mu; \mathbb{R}) \), with \( \mu \) a finite (unsigned) measure.
Let \( \phi \in (L^p)^* \). Find \( g \in L^q \) s.t. \( L_g = \phi \).
For each measurable \( E \), \( \mu(E) \) is finite and so \( \chi_E \in L^p \).
Define a finite signed measure on \( \Omega \) by

\[ \nu(E) = \phi(\chi_E) \]

Verify that \( \nu \) really is a signed measure e.g. if \( E_1, E_2, \ldots \) are measurable and disjoint, then

\[ \nu(\bigcup E_i) = \phi(\chi_{\bigcup E_i}) = \phi(\sum \chi_{E_i}) = \sum \phi(\chi_{E_i}) = \sum \nu(E_i) \]

Here, we are using linearity and continuity of \( \phi \).

If \( \mu(E) = 0 \), then \( \chi_E = 0 \) in \( L^p \) and so \( \nu(E) = 0 \).
Thus, \( \nu << \mu \).

By Radon-Nikodym, there is a measurable function \( g \) s.t. for all measurable \( E \)

\[ \phi(\chi_E) = \nu(E) = \int_E g d\mu \]

Extend to simple functions \( f \in L^p \) to get

\[ \phi(f) = \int f g d\mu \]
Extend to $L^p$ functions via density of simple function in $L^p$ (Prop. 6.7, proven by Stephen G. in Math 420) and continuity of $\phi$.

So, for all $f \in L^p$,

$$\phi(f) = \int fg\,d\mu$$

We claim that $g \in L^q$.

Proof: (Folland, Theorem 6.14):