Lecture 12:

Recall:

Real HB Theorem: Let W be a subspace of X, a real vector spa. Let $f : W \to \mathbb{R}$ be a linear functional. Let $p : X \to \mathbb{R}$ be a sublinear functional such that for all $x \in W$, $f(x) \leq p(x)$. Then f can be extended to a linear functional F on X s.t. $F(x) \leq p(x)$ for all $x \in X$ and $F|_W = f$.

Prop 1: Let X be a complex vector space.

a. Let f be a $\mathbb{R}$-linear functional.

Then $f_C(x) := f(x) - if(ix)$ is a $\mathbb{C}$-linear functional, the complexification of f.

b. Let h be a $\mathbb{C}$-linear functional and let $f = \Re h$.

Then f is a $\mathbb{R}$-linear functional and $h(x) = f_{\mathbb{C}}(x)$.

$$\begin{array}{ccc} \mathbb{R} & & \mathbb{C} \\ f = \Re h & \Leftrightarrow & h = f_{\mathbb{C}} \end{array}$$

Prop 2a: Let f be a $\mathbb{R}$-linear functional.

Let $p(x)$ be a $\mathbb{C}$-seminorm on X. Then

$$\forall x \ |f(x)| \leq p(x) \Leftrightarrow \forall x \ |f_{\mathbb{C}}(x)| \leq p(x)$$

Complex Hahn-Banach Theorem: Let X be a complex vector space and $p(x)$ a $\mathbb{C}$-semi-norm on X.

Let W be a complex subspace of X.

Let h be $\mathbb{C}$-linear-functional on W s.t, $\forall x \ |h(w)| \leq p(w)$.

Then there is a $\mathbb{C}$-linear-functional H on X s.t.
$H|_W = h$ and $\forall x \ |H(x)| \leq p(x)$.

$$
\begin{array}{cccc}
 & \mathbb{R} & & \mathbb{C} \\
X & F & \rightarrow & H = F_{\mathbb{C}} \\
 & \uparrow & & \\
W & f = \Re h & \leftarrow & h
\end{array}
$$

Proof of Complex HB:

Let $f = \Re h$, and so by Prop 1b, $h = f_{\mathbb{C}}$.

Then f is a $\mathbb{R}$-linear functional on W s.t. $\forall w \in W,\ f(w) \leq |f(w)| \leq |h(x)| \leq p(w)$.

Since p is a $\mathbb{C}$-semi-norm, it is a $\mathbb{R}$-semi-norm and hence a semi-linear functional.

By real HB, there is a $\mathbb{R}$-linear functional F on X s.t. $F|_W = f$ and $\forall x \in X$, $F(x) \leq p(x)$.

But then for all $\forall x \in X$ $|F(x)| \leq p(x)$ (when $F(x)$ is negative, apply $F(x) \leq p(x)$ to $-x$).

Let $H := F_{\mathbb{C}} = F(x) - iF(ix)$. Then,

$$
H|_W(x) = (F(x) - iF(ix))|_W = f(x) - if(ix) = f_{\mathbb{C}}(x) = h(x).
$$

And by Prop. 2a, since $\forall x \in X$ $|F(x)| \leq p(x)$, we have $\forall x \in X$ $|H(x)| \leq p(x)$. $\square$

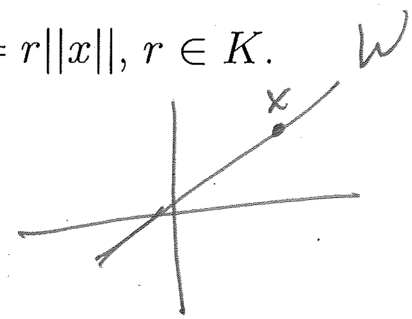
Theorem ((Folland Theorem 5.8b,c)) Consequences of HB Theorem

1. Let $x \in X$ s.t. $x \neq 0$. There exists a BLF f on X s.t. $||f|| = 1$ and $f(x) = ||x||$.
2. The BLFs separate points in X, i.e., for $x, y \in X$, there exists a BLF f on X s.t. $f(x) \neq f(y)$.

So, there are lots of BLFs.

Proof:

1: Define g on $W := Kx$ by $g(rx) = r||x||$, $r \in K$.
Then for all $w \in W$, $|g(w)| = ||w||$.



Apply H-B with $p(u) = ||u||$, to get a BLF f s.t.

$f|_W = g$ and $|f(u)| \leq ||u||$ for all $u \in X$.

So, $||f|| \leq 1$. But $f(x) = g(x) = ||x||$, and so $||f|| = 1$.

2. Let $x, y \in X, x \neq y$. Let f be a BLF as in part 1 s.t. $f(x-y) = ||x-y|| \neq 0$. Then $f(x) \neq f(y)$.

Time out: Isomorphisms in different categories of math:

Isomorphism is a map that is

1. Bijective (or nearly bijective)

2. Preserves a structure

3. Inverse preserves structure

Sometimes, 3 is automatic.

1. Defn: A *vector space isomorphism* is a bijective linear map from one vector space X to another Y, with linear inverse.

Fact: A bijective linear map automatically has a linear inverse and thus is a vector space isomorphism.

Proof: Let $T : X \to Y$ be a bijective linear transformation.

Let $y_1, y_2 \in Y, a_1, a_2 \in K$ and T^{-1} be the inverse of T as a set mapping. Let

$$x = T^{-1}(a_1y_1 + a_2y_2), x_1 = T^{-1}(y_1), x_2 = T^{-1}(y_2)$$

Then

$$T(x) = a_1y_1 + a_2y_2 = a_1T(x_1) + a_2T(x_2) = T(a_1T^{-1}(y_1) + a_2T^{-1}(y_2))$$

the latter equality because T is linear. Since T is injective, we get

$$T^{-1}(a_1y_1 + a_2y_2) = x = (a_1T^{-1}(y_1) + a_2T^{-1}(y_2)$$

So, T^{-1} is linear. $\square$

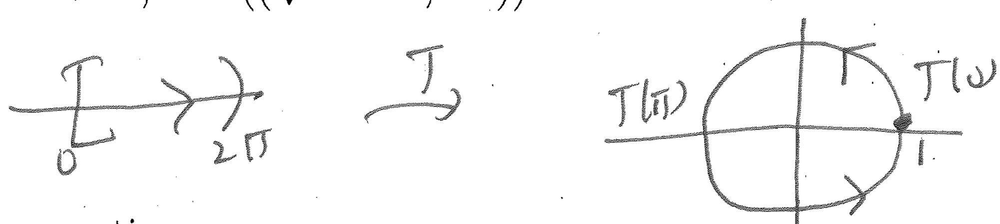
2. Defn: A *homeomorphism* is a bijective continuous map from one metric space X (more generally, topological space) to another Y, with continuous inverse.

Fact: A bijective continuous map need not have a continuous inverse and thus need not be a homeomorphism.

Example:

$$T : [0, 2\pi) \to S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\},\ T(\theta) = (\cos(\theta), \sin(\theta))$$

T is bijective because it maps onto the unit circle and $(\cos(\theta), \sin(\theta))$ uniquely determine θ in $[0, 2\pi)$. T is continous because $\cos(\theta)$ and $\sin(\theta)$ are continuous. But T^{-1} is not continuous, since $T^{-1}(1, 0) = 0$ and for $\epsilon > 0$, $T^{-1}((\sqrt{1 - \epsilon^2}, -\epsilon))$ is close to 2π.



An exception:

(Math 320) A bijective continuous map from one compact metric space to another automatically has a continuous inverse and therefore is a homeomorphism.

3. Defn: A *homeomorphic isomorphism* (isomorphism) is a bijective continuous linear map from one NVS to another, with a continuous inverse.

Fact: A bijective continuous linear map need not have a continuous inverse and so need not be a homeomorphism isomorphism:

Example in HW2 (4d). A Bijective BLT with whose inverse is not a BLT.

An exception:

Corollary of Open Mapping Theorem: A bijective continuous linear map from one *Banach space* to another automatically has a continuous inverse and therefore is a homeomorphic isomorphism.

4. Notion of Sameness for NVS:

Defn: An *isometric isomorphism* is a bijective norm-preserving linear map from one NVS to another that has a norm-preserving inverse.

Fact: A bijective norm-preserving linear map automatically has a norm-preserving inverse:

You effectively need to prove this in Hw2, 6a.

Note: In HW, I defined isometric isomorphism, without the norm-preserving inverse.

Isometric isomorphism:
1. Linear
2. Norm-preserving
3. Injective
4. Surjective

Note: 2 implies 3: if $\Psi(x) = 0$, then $||x|| = ||\Psi(x)|| = ||0|| = 0$ and so Ψ has trivial kernel.

Also, sometimes we refer to an isometric isomorphism onto its image, so we don't need to prove 4.

Lecture 13:

Defns: The *dual space* of an NVS X is defined

$$X^* = L(X, K) = \{BLFs\ f : X \mapsto K\}$$

Note that X^* is a Banach space, with the operator norm topology, even if X is not, because "Y is complete" implies that "$L(X, Y)$ is complete."

Defn: The *double dual space* of an NVS X is

$$X^{**} = (X^*)^*$$

For $x \in X$, define $\hat{x} \in X^{**}$ by

$$\hat{x}(f) = f(x)$$

Prop: $\hat{x} \in X^{**}$, i.e. $\hat{x}$ is a BLF on X^*.
Proof: $\hat{x}$ a linear functional:

$$\hat{x}(af + bg) = (af + bg)(x) = af(x) + bg(x) = a\hat{x}(f) + b\hat{x}(g).$$

Since

$$|\hat{x}(f)| = |f(x)| \leq ||f||\ ||x|| = ||x||\ ||f||$$

we have

$$||\hat{x}|| \leq ||x||.$$

and so $\hat{x}$ is a BLF on X^*. $\square$

In fact, Prop: $||\hat{x}|| = ||x||$.

Proof: Enough to find f s.t. $|\hat{x}(f)| = ||x||\ ||f||$.

By Theorem above (Folland 5.8b), there exists a linear functional f on X s.t. $||f|| = 1$ and $f(x) = ||x||$. So,

$$|\hat{x}(f)| = |f(x)| = ||x|| = ||x||\ ||f||$$

and so $||\hat{x}|| = ||x||$. □

Note: $||\hat{x}|| = ||\hat{x}||_{X^{**}}, ||x|| = ||x||_X$.

Defn: The *canonical embedding* is defined:

$$X \to X^{**}, x \mapsto \hat{x}$$

Theorem (Folland Theorem 5.8d): The canonical embedding $X \to X^{**}$, $x \mapsto \hat{x}$ is an isometric isomorphism onto its image Y.

X and Y are the "same,"

Proof:

Linearity: for all $f \in X^*$,

$$\widehat{ax+by}(f) = f(ax+by) = af(x)+bf(y) = a\hat{x}(f)+b\hat{y}(f) = (a\hat{x}+b\hat{y})(f)$$

Norm-preserving: as mentioned above, $||\hat{x}|| = ||x||$.

Injective: as mentioned above, 1-1 follows from norm-preserving.

□

Proposition: A NVS is Banach iff the image of its canonical embedding is closed in X^{**}.

Proof: X^{**} is Banach since it is the dual of X^*.

So, the image Y of the NVS X is closed iff Y is complete iff X is complete, the latter "iff" since X and Y are isometrically isomorphic and so one is complete iff the other is complete □

Note that in any event an NVS embeds as a dense subspace of the closure $\overline{Y}$ of the image Y of canoncial embedding, which is a Banach space. Deja vu?

Defn: An NVS is *reflexive* if the image of the canonical imbedding is X^{**}. We sometimes write this as $X = X^{**}$, having identifed X with its image.

Note that a reflexive NVS is automatically a Banach space because the image of its canonical embedding is X^{**}, which is trivially closed.

Alternatively, X^{**} is Banach because it is a dual space.

Reflexive Banach spaces are rare but have lots of nice properties.

Example of a dual space:

Theorem (Folland Thm 6.15) Let $1 < p < \infty, 1/p + 1/q = 1$. Then, $(L^p)^* = L^q$. More precisely, $(L^p)^*$ and L^q are isometrically isomorphic.

Corollary $(L^p)^{**} = (L^q)^* = (L^p)$, and so L^p is reflexive.

Sktech of proof of: $(L^p)^* = L^q$:

Let $g \in L^q$.

$$L_g : L^p \to \mathbb{R}, L_g(f) = \int fg d\mu$$

Recall (from awhile back): L_g is a BLF (and thus $L_g \in (L^p)^*$) and $||L_g|| = ||g||_q$.

Let $\Psi : L^q \to (L^p)^*,\ g \mapsto L_g$

Proposition: Ψ is an isometric isomorphism.

So, $(L^p)^* = L^q$,

Proof of Prop:
Linearity: $g \mapsto L_g$ is linear:

$$L_{ag_1+bg_2}(f) = \int (ag_1 + bg_2)(f) d\mu = a \int g_1 f d\mu + b \int g_2 f d\mu$$
$$= aL_{g_1}(f) + bL_{g_2}(f)$$

and so

$$L_{ag_1+bg_2} = aL_{g_1} + bL_{g_2}$$

Norm-preserving: $||L_g|| = ||g||_q$.

Injective: as noted above, injecttive follows from norm-preserving.

Surjective: The hard part (Theorem 6.15) Rough idea:

Say $L^p = L^p(\Omega, \mu; \mathbb{R})$, with μ a finite (unsigned) measure.

Let $\phi \in (L^p)^*$. Find $g \in L^q$ s.t. $L_g = \phi$.

For each measurable E, $\mu(E)$ is finite and so $\chi_E \in L^p$.

Define a finite signed measure on Ω by

$$\nu(E) = \phi(\chi_E)$$

Verify that ν really is a signed measure e.g. if $E_1, E_2, \ldots$ are measble and disjoint, then

$$\nu(\cup E_i) = \phi(\chi_{\cup E_i}) = \phi(\sum \chi_{E_i}) = \sum_i \phi(\chi_{E_i}) = \sum_i \nu(E_i)$$

Here, we are using linearity and continuity of ϕ.

If $\mu(E) = 0$, then $\chi_E = 0$ in L^p and so $\nu(E) = 0$.

Thus, $\nu << \mu$.

By Radon-Nikodym, there is a measurable function g s.t. for all measble E

$$\phi(\chi_E) = \nu(E) = \int_E g d\mu$$

Extend to simple functions $f \in L^p$ to get

$$\phi(f) = \int f g d\mu$$

Extend to L^p functions via density of simple function in L^p (Prop. 6.7, proven by Stephen G. in Math 420) and continuity of ϕ.

So, for all $f \in L^p$,

$$\phi(f) = \int fgd\mu$$

We claim that $g \in L^q$. .

Proof: (Folland, Theorem 6.14):