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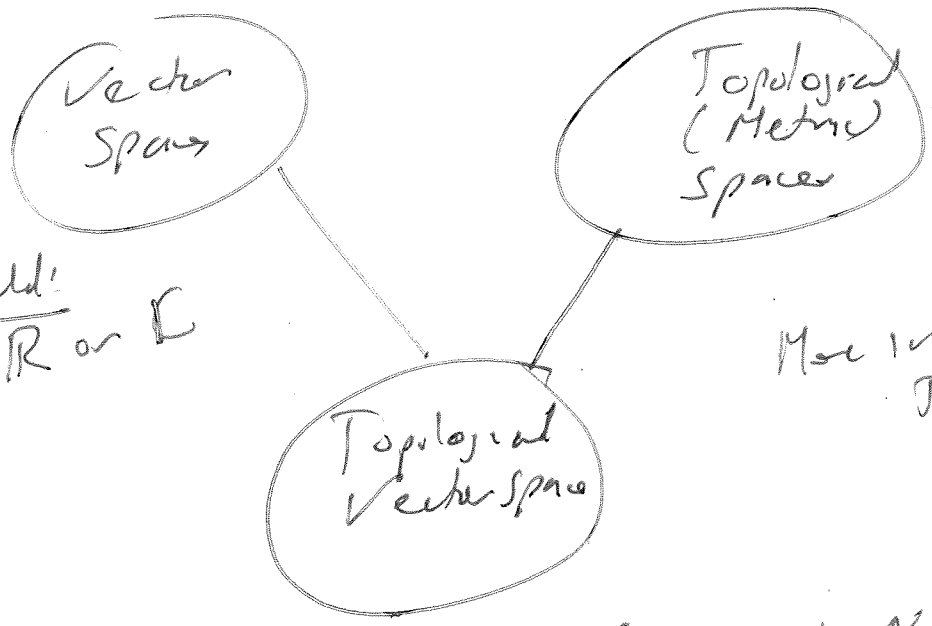
Math 510/421 : Functional Analysis

Functional Analysis: Study of Topological Vector Space (TVS)

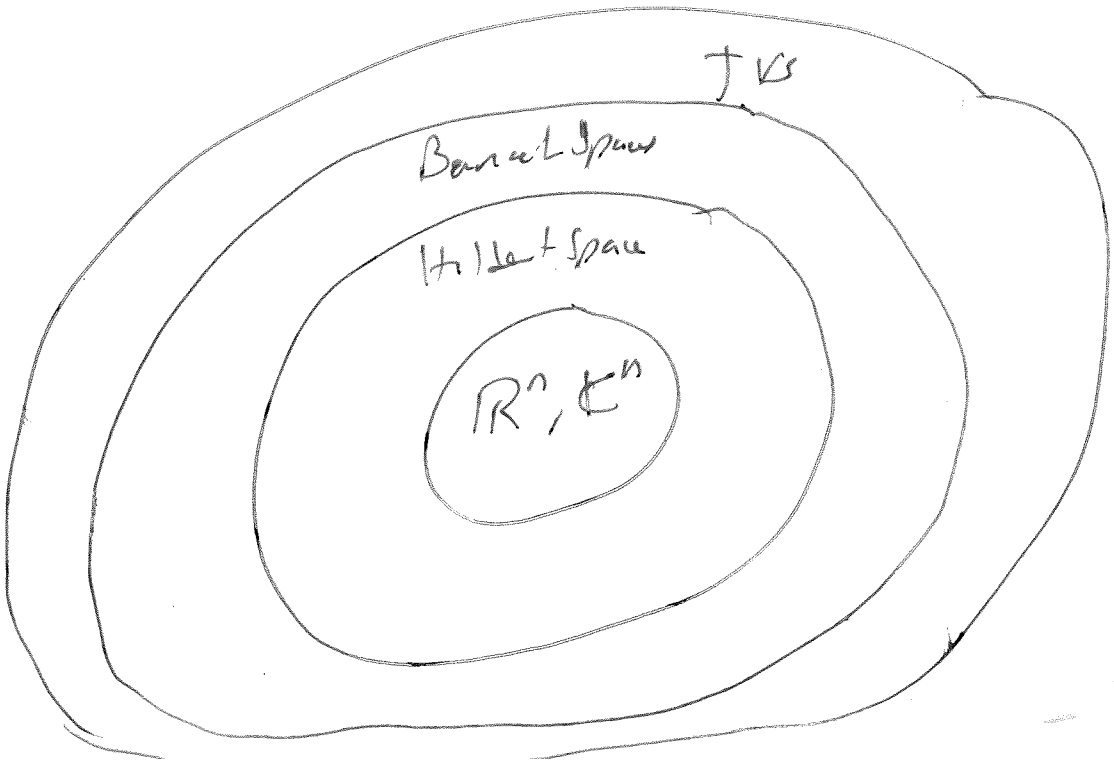


Base Field: $K = \mathbb{R}$ or $\mathbb{C}$

More intuitive on Top Space

Main Classes: Banach Space (Complete Normed Vector Space)

Hilbert Space (Complete Inner Product Space)



Introduction:

Functional analysis is the study of *topological vector spaces*, i.e., vector spaces with a topology (metric) under which the vector space operations are continuous.

This is interesting when vector space is infinite dimensional. The same space can have interesting but very different topologies. ~~In the finite dimensional case, all the reasonable topologies coincide~~.

Base field for vector space: $K = \mathbb{R}$ or $\mathbb{C}$.

Main examples are *Banach spaces* (complete normed vector spaces) and *Hilbert spaces* (complete inner product spaces). metric

The universe of functional analysis:

$\mathbb{R}^n, \mathbb{C}^n \subset$ Hilbert spaces $\subset$ Banach spaces $\subset$ Topological vector spaces

These kinds of spaces grew out of axiomatizing the kinds of spaces of functions that arose in analysis: Fourier theory (limits of linear combinations of sin's and cos's and complex exponentials) and differential equations (spaces of solutions of differential equations). 2

We will study the structure of these spaces, with respect to different meaningful topologies, and the continuous linear transformations from one such space to another. 1

Applications to most areas of modern analysis including pdes, harmonic analysis, probability, operator theory, ergodic theory, convex analysis and optimization, as well as to areas of physics such as quantum mechanics. 3

Pre-req: Measure Theory (Math 420/507).

– Assume also a solid undergrad course in real analysis in metric spaces (convergence, continuity, open, closed, compact, Cauchy, complete) like 320-321

– linear algebra (vector spaces, linear independence, basis, dimension, linear transformations).

– ideally, a bit of general topology (chapter 4 of Folland; we will introduce bits as we need them).

Textbook: Folland, Real Analysis, 2nd edition

Course Outline:

5.1: Banach spaces, including L^p spaces (6.1); norm determines a metric.

5.2: Linear functionals on Banach spaces; Hahn-Banach theorem and corollaries: any Banach space has lots of norm-continuous linear functionals $f : X \to K$. *Dual space:*

$X^* = \{$ norm-cts linear functionals on $X\}$.

5.5: Hilbert spaces – has many properties of finite diml. spaces.

5.3: Open mapping theorem, closed graph theorem and uniform boundedness principle, based on Baire Category Theorem

5.4: Topological vector spaces, weak topologies on Banach sapces and

Banach-Alaoglu theorem: The unit ball is weakly compact.

(In an infinite-dimensional normed vector space, the unit ball is never norm-compact)

Weak convergence of Borel probability measures.

(Riesz) representation Theorems : characterization of dual spaces of certain Banach spaces (three different versions (Theorems 5.25, 6.15, 7.17)).

Convexity and Krein-Milman Theorem: under suitable hypotheses, "convex set is convex hull of its extreme points"

Ergodic theory: stuedy of Measure-preserving transformations (MPTs)

Evaluation: 50% bi-weekly homework and 50% Final Exam. First HW: due Friday, Jan. 18.

Conventions:

– All measure spaces $(X, \mathcal{B}, \mu)$, often written as (X, μ), are assumed sigma-finite. Many will be finite. Some will be probability measures.

– Terminology: use *Vector space* and *Linear space* interchangeably.

Webpage
Office Hours MWF 1:30-2:30
Sign-up Sheet

Review of Metric spaces:

Defn: *metric space* (X, d) where X is a set and d is a *metric* $d(x, y) \geq 0$, i.e., satisfies

- $d(x, x) = 0$ and $d(x, y) = 0$ iff $x = y$.
- $d(y, x) = d(y, x)$
- $d(x, z) \leq d(x, y) + d(y, z)$

Metric measures distance.

Main example: $\mathbb{R}^n$ or $\mathbb{C}^n$ with *Euclidean metric*, $d(x, y) = \sqrt{\sum_i (|x_i - y_i|^2)}$.

On $\mathbb{R}$ or $\mathbb{C}$, this reduces to $d(x, y) = |x - y|$.

Another example: *Discrete metric* on any set X:

$$d(x, y) = \left\{ \begin{array}{ll} 1 & x \neq y \\ 0 & x = y \end{array} \right\}$$

Convergence: $x_n \to x$ means $d(x_n, x) \to 0$ ("x is the limit of x_n") i.e., if for all $\epsilon > 0$, there exists $N = N(\epsilon)$ s.t. if $n \geq N$, then $d(x_n, x) < \epsilon$.

Cauchy: x_n is *Cauchy* if for all $\epsilon > 0$, there exists N s.t. if $m, n \geq N$, then $d(x_n, x_m) < \epsilon$.

Defn: a metric space is *complete* if every Cauchy sequence converges (to a point in X).

Example: $\mathbb{R}^n$ and $\mathbb{C}^n$ with Euclidean metric is complete. $(0, 1)$ and Q with Euclidean metric are not complete;

Two easy results, but very important:

Prop: Every convergent sequence is Cauchy.

Part

Proof: Let $x_n \to x$. Given $\epsilon > 0$, choose N s.t. if $n \geq N$, then $d(x_n, x) < \epsilon$.

if $n, m \geq N$, then

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < 2\epsilon$$

$\square$

Prop: If a subsequence of a Cauchy sequence converges, then the sequence converges (to the same limit).

Proof: Let x_n be a Cauchy sequence and x_{n_i} a subsequence which converges to x.

Let $\epsilon > 0$. Choose I s.t. if $i \geq I$, then $d(x_{n_i}, x) < \epsilon$.

Choose N s.t. if $m, n \geq N$, then $d(x_n, x_m) < \epsilon$.

Choose $i \geq I$ s.t. $n_i \geq N$. Then for $m \geq N$,

$$d(x_m, x) \leq d(x_m, x_{n_i}) + d(x_{n_i}, x) < 2\epsilon.$$

$\square$

Notation: Open and closed balls:

$$B_\epsilon(x) = \{y \in X : d(x, y) < \epsilon\}, \cancel{\overline{B}_\epsilon(x) = \{y \in X : d(x, y) \leq \epsilon\}}$$

~~Caution: In a metric space, $\overline{B}_\epsilon(x)$ is not always the closure of $B_\epsilon(x)$ but this is true for NVS.~~

Defns: For subsets of a metric space,

U is *open* if for all $x \in U$, there exists $\epsilon > 0$ s.t. $B_\epsilon(x) \subset U$. Equivalently, U is a union of open balls.

F is *closed* if it contains all its limit points. Equivalently, F^c is open.

C is *compact* if every sequence has a convergent subsequence. Equivalently, every open cover has a finite subcover.

Cpct $\Rightarrow$ closed & bounded

Caution: there are metric spaces in which compact sets need not be closed and bounded.

$f : X \to Y$ is *continuous*: usual δ, ϵ definition. Equivalently, $f : X \to Y$ is continuous iff for all open sets U in Y, $f^{-1}(U)$ is open.

Exercises relating completeness, closedness, compactness, continuity:

1. A subset of a complete metric space is complete iff it is closed.
2. Every compact metric space is complete.
3. For a subset of a compact metric space X, TFAE:

   $S \subseteq X$

   - S is complete
   - S is closed
   - S is compact.
4. Uniform limit of continuous functions is continuous.

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Lecture 2:

5-minute intro to Topological Spaces (Folland, Ch. 4)

Defn: A *topological space* is a pair $(X, \mathcal{T})$ where X is a set. $\mathcal{T} \subset P(X)$ is a *topology*, i.e., satisfies:

- $\emptyset, X \in \mathcal{T}$
- $\mathcal{T}$ is closed under arbitrary unions
- $\mathcal{T}$ is closed under finite intersections

Main Example: (X, d) is a metric space and $\mathcal{T}$ is the collection of open sets, i.e., arbitrary unions of open balls.

Exercise: show that this $\mathcal{T}$ is indeed a topology.

The idea is that a topology $\mathcal{T}$ abstracts the notion of "open set."

The topology of the Euclidean metric is the collection of usual open sets in $\mathbb{R}^n$.

Exercises:

a. the topology of the discrete metric is the collection of all subsets

b. the trivial topology, $\mathcal{T} = \{\emptyset, X\}$, on a set with at least two points does not come from a metric.

There will be much less trivial and much more important topologies that do not come from metrics.

~~Cantor's Topological Concepts: open, closed, cpct, connected, continuous~~

~~Metric~~ Completeness

Normed Vector Spaces (NVS) and Banach spaces

Defn: A *norm* on a vector space X over a field $K = \mathbb{R}$ or $\mathbb{C}$ is a function $X \to \mathbb{R}$, $x \mapsto ||x||$ that satisfies

- Positivity: $||x|| \geq 0$; and $||x|| = 0$ iff $x = 0$.
- Homgeneity: $||\lambda x|| = |\lambda| ||x||$
- Subadditivity: $||x + y|| \leq ||x|| + ||y||$

Defn: A *normed vector space* (NVS) (or normed linear space) is a vector space together with a norm, $||(X, ||\cdot||)$.

The standard norms on $\mathbb{R}^n$ and $\mathbb{C}^n$are indeed norms:

- Euclidean norm $||x||_2 = \sqrt{\sum_i |x_i|^2}$
- Sup norm: $||x||_{\text{sup}} = \max_i |x_i|$
- Manhattan norm: $||x||_1 = \sum_i |x_i|$

It is easy to check that each is a norm. For instance, $||x||_{\text{sup}}$ clearly satisfies positivity and homogeneity. For the triangle inequality:

$$||x + y||_{\text{sup}} = \max_i |x_i + y_i| \leq \max_i (|x_i| + |y_i|)$$

$$\leq \max_i (||x||_{\text{sup}} + ||y||_{\text{sup}}) = ||x||_{\text{sup}} + ||y||_{\text{sup}}$$

Any norm determines a metric: $d(x, y) := ||x - y||$:

- $d(x, x) = ||x - x|| = ||0|| = 0$ and $d(x, y) = 0$ iff $||x - y|| = 0$ iff $x = y$.
- $d(y, x) = ||y - x|| = ||(-1)(x - y)|| = |-1| ||(x - y)|| = ||(x - y)||$
- $d(x, z) = ||x - z|| \leq ||x - y|| + ||y - z|| = d(x, y) + d(y, z)$

So a NVS can also be considered as a metric space. This metric is called the *norm metric* and the topology is called the *norm topology*.

$$B_\epsilon(x) = \{y \in X : ||y - x|| < \epsilon\}, \overline{B_\epsilon(x)} = \{y \in X : ||y - x|| \leq \epsilon\}$$

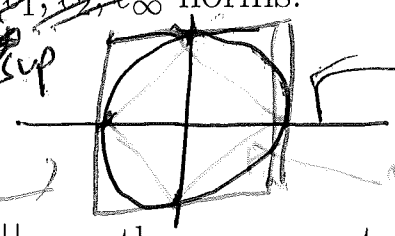
Exercise: For an NVS, as the notation suggests, ~~the~~ $\overline{B_\epsilon(x)}$ is the closure of $B_\epsilon(x)$ (in contrast to general metric spaces).

Of special interest with be the open unit ball, closed unit ball and unit sphere

$$B_1 = B_1(0), \overline{B}_1 = \overline{B}_1(0), S_1 = \{x \in X : ||x|| = 1\},$$

Pictures of unit spheres in each ~~$\ell_1, \ell_2, \ell_\infty$~~ norms.

$||\cdot||_1, ||\cdot||_2, ||\cdot||_{sup}$



$||\cdot||_{sup}$

Defn: Two norms $||\cdot||_1, ||\cdot||_2$ on the same vector space X are *equivalent* if there exist $C_1, C_2 > 0$ s.t. for all $x \in X$

$$C_1||x||_1 \leq ||x||_2 \leq C_2||x||_1$$

i.e., ratios of 1-norm and 2-norm are uniformly bounded above and below.

The metrics for two equivalent norms have the same convergent sequences, Cauchy sequences, and one is complete as a metric space iff the other is

– because for a sequence x_n,

$$||x_n - x_m||_2 \leq C_2||x_n - x_m||_1 \text{ and } ||x_n - x_m||_1 \leq (1/C_1)||x_n - x_m||_2$$

Exercise: Two norms on the same vector space have the same topologies, i.e., same collections of open sets, iff they are equivalent as norms. So, if two norms have the same norm topologies, then one is complete iff the other is (this is false for metrics in general).

Exercise: on a finite-dimensional vector space any two norms are equivalent as norms (and therefore have the same norm topology), e.g., Manhatttan, sup and euclidean norms are all equivalent.

We will see that this is far form true on infinite-dimensional spaces, and this is what makes functional analysis interesting.

Examples of Banach spaces:

Defn: A *Banach space* is a complete NVS.

Will introduce properties of Banach spaces as needed.

1. $\mathbb{R}^n$ and $\mathbb{C}^n$ with any norm. Since any two norms are equivalent, it suffices to check that the Euclidean metric is complete, which it is.

2. Let Ω be ANY set.

$$B(\Omega) = \{bounded \;\; f : \Omega \to K\}$$

with sup *norm*: for $f : X \to K$,

$$||f||_{sup} := \sup\{|f(x)| : x \in \Omega\},$$

Then $(B(\Omega), ||\cdot||_{sup})$ is a Banach space.

For definiteness, use $K = \mathbb{R}$.

Special case: $B(\mathbb{N})$, often called ℓ_∞, is the set of all bounded sequences $\{a_1, a_2, \ldots, a_n, \ldots,\}$

Check: a. vector space, b. norm, c. complete

a. $\mathbb{R}^\Omega = \{f : \Omega \to \mathbb{R}\}$ is a vector space.

To show that $B(\Omega)$ is a vector space, we need only show that it is a subspace of $\mathbb{R}^\Omega$, equivalently that it is closed under vector addition and scalar multipliation:

– vector addition: addition of functions $(f+g)(x) = f(x)+g(x)$; if f and g are bounded, then so is $f+g$.

– scalar multiplication: $(\lambda f)(x) = \lambda f(x)$; if f is bounded, then so is λf.

b. $||f|| = ||f||_{sup}$ is a norm:

– Positivity: clearly $||f|| \geq 0$ and $||f|| = 0$ iff $f = 0$.