Lecture 37:

*Review since last Review:*

**BCT:** In a complete metric space $X$,
- a countable intersection of open dense sets is dense
- $X$ cannot be written as a countable union of nowhere dense sets

**Residual:** contains a countable intersection of open dense sets

**Meager:** a countable union of nowhere dense sets

Meager and Residual sets are complementary

**Hamel bases** (finite linear combinations): use BCT to show that for a Banach space a Hamel basis is either finite or uncountable

**Schauder bases:** uses infinite linear combinations;
- Schauder implies separable

**Open Mapping Theorem:** Banach to Banach, surjective BLT; then BLT is open.
- Main corollary Banach to Banach, bijective BLT; then BLT is a homeo.

**Closed Graph Theorem:** Banach to Banach; a linear transformation has a closed graph iff it is continuous.
- Harder part (uses Open mapping theorem) is “closed implies continuous”

**Uniform Boundedness Principle:** $X$ Banach, $Y$ NVS, $T_n$ BLT; If for each $x$ $\{T_n x\}$ is bounded, then $\{|T_n||\}$ is bounded.

**Topological Vector Spaces:** Marriage of topology and linear algebra
- vector addition and scalar multiplication are continuous

Norm Topology on NVS $X$:
- $x_n \to x$: $\|x_n - x\| \to 0$
- nhbhd base at $x$: $\{B_r(x)\}$

Weak Topology:
- weakest topology on NVS $X$ s.t. all BLFs are cts
- $x_n \to w x$: for all $f \in X^*$, $f(x_n) \to f(x)$
- nhbd base at $x$: finite intersections of $U_{f,x,\epsilon} = \{y \in X : |f(y) - f(x)| < \epsilon\}$

- weak closure of unit sphere is closed unit ball
- subspaces are weakly closed iff norm closed

Weak* Topology:
- weakest topology on $X^*$ s.t. all $\hat{x}$, $x \in X$ are continuous
- $f_n \to wk^* f$: for all $x \in X$, $f_n(x) \to f(x)$, equivalently for all $x \in X$, $\hat{x}(f_n) \to \hat{x}(f)$,
- nhbd base at $f$: finite intersections of $U_{x,f,\epsilon} = \{g \in X^* : |g(x) - f(x)| < \epsilon\}$

On $X^*$

$\mathcal{T}_{wk^*} \subset \mathcal{T}_{wk} \subset \mathcal{T}_{norm}$

Banach-Alouglu Theorem: unit ball in $X^*$ is weak* compact.

Riesz Representation Theorem for $C(\Omega)$, $\Omega$ a compact metric space:
- $C(\Omega)^* = N(\Omega)$, collection of finite signed measures on $\Omega$
\[ M_T(\Omega) \subset M(\Omega) \subset B(\Omega)^* \subset N(\Omega) \]

where

- \( B(\Omega)^* \) is unit ball in \( N(\Omega) \)
- \( M(\Omega) \) = collection of Borel probability measures on \( \Omega \)
- \( M_T(\Omega) \) = collection of \( T \)-invariant Borel probability measures \( \mu \) on \( \Omega \), where \( T : \Omega \to \Omega \) is cts and for all Borel \( A \), \( \mu(T^{-1}(A)) = \mu(A) \).

\( M_T(\Omega) \) is nonempty, weak* compact, convex sitting in \( N(\Omega) \) locally convex Hausdorff TVS.

An **extreme point** is a point in a convex set that cannot be written as a non-trivial convex combination of two distinct points in the set; here, non-trivial means \( c = ta + (1 - t)b, a \neq b, 0 < t < 1, \) i.e., \( c \in \text{int}[a, b] \).

Let \( X \) be a vector space and \( A \subset X \). The **closed convex hull** of \( A \) is the intersection of all closed convex sets containing \( A \), equivalently the closure of the intersection of all convex sets containing \( A \).

In some sense, the closed convex hull of \( A \) is the set of all limits of convex combinations of elements of \( A \). So it is the set of all limits of weighted averages of elements of \( A \).

**Krein-Milman Theorem**: Let \( X \) be a locally convex Hausdorff Topological Vector Space. Let \( K \subset X \) be a compact convex set. Then \( K \) is the closed convex hull of its extreme points.

**Application to**: \( M_T(\Omega) \):

**Defn**: An MPT \( T \) is **ergodic** w.r.t. \( \mu \) if for all \( A \in \mathcal{A} \) whenever \( T^{-1}(A) = A \), then \( \mu(A) \) is 0 or 1.

**Examples**: Doubling map and irrational rotations.
Ergodic Theorem: An MPT $T$ is ergodic iff for all $f \in L^1(\mu)$, for $\mu$-a.e. $x \in \Omega$

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x) = \int f \, d\mu$$

“Time average = space average”

Theorem: Let $T$ be continuous on a compact metric $\Omega$. Then $T$ is ergodic w.r.t $\mu \in M_T(\Omega)$ iff $\mu$ is an extreme point of $M_T(\Omega)$.

So, Krein-Milman Theorem implies:
— the existence of an ergodic invariant measure in $M_T(\Omega)$ for any continuous $T$.
— a decomposition of invariant measures as “averages” (limits of convex combinations) of ergodic invariant measures.

THE END