Lecture 37:

Review since last Review:

BCT: In a complete metric space X,

– a countable intersection of open dense sets is dense

– X cannot be written as a countable union of nowhere dense sets

Residual: contains a countable intersection of open dense sets

Meager: a countable union of nowhere dense sets

Meager and Residual sets are complementary

Hamel bases (finite linear combinations): use BCT to show that for a Banach space a Hamel basis is either finite or uncountable

Schauder bases: uses infinite linear combinations;

– Schauder implies separable

Open Mapping Theorem: Banach to Banach, surjective BLT; then BLT is open.

– Main corollary Banach to Banach, bijective BLT; then BLT is a homeo.

Closed Graph Theorem: Banach to Banch; a linear transformation has a closed graph iff it is continuous.

– Harder part (uses Open mapping theorem) is "closed implies continuous"

Uniform Boundedness Principle: X Banach, Y NVS, T_n BLT; If for each x $\{T_n x\}$ is bounded, then $\{||T_n|||\}$ is bounded.

Topological Vector Spaces: Marriage of topology and linear algebra

– vector addition and scalar multiplication are continuous

Norm Topology on NVS X:

– $x_n \to x$: $||x_n - x|| \to 0$

– nhbhd base at x: $\{B_r(x)\}$

Weak Topology:

– weakest topplogy on NVS X s.t. all BLFs are cts

– $x_n \stackrel{w}{\to} x$: for all $f \in X^*$, $f(x_n) \to f(x)$

– nbhd base at x: finite intersections of

$$U_{f,x,\epsilon} = \{y \in X : |f(y) - f(x)| < \epsilon$$

– weak closure of unit sphere is closed unit ball

– subspaces are weakly closed iff norm closed

Weak* Topology:

– weakest topology on X^* s.t. all $\hat{x}$, $x \in X$ are continuous

– $f_n \stackrel{wk*}{\to} f$: for all $x \in X$, $f_n(x) \to f(x)$, equivalently for all $x \in X$, $\hat{x}(f_n) \to \hat{x}(f)$,

– nbhd base at f: finite intersections of

$$U_{x,f,\epsilon} = \{g \in X^* : |g(x) - f(x)| < \epsilon$$

On X^*

$$\mathcal{T}^{wk*} \subset \mathcal{T}^{wk} \subset \mathcal{T}^{norm}$$

Banach-Alouglu Theorem: unit ball in X^* is weak* compact.

Riesz Representation Theorem for $C(\Omega)$, Ω a compact metric space:

– $C(\Omega)* = N(\Omega)$, collection of finite signed measures on Ω

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$$M_T(\Omega) \subset M(\Omega) \subset B(\Omega)^* \subset N(\Omega)$$

where

—- $B(\Omega)^*$ is unit ball in $N(\Omega)$

—- $M(\Omega)$ = collection of Borel probability measures on Ω

—- $M_T(\Omega)$ = collection of T-invariant Borel probability measures μ on Ω, where $T : \Omega \to \Omega$ is cts and for all Borel A, $\mu(T^{-1}(A)) = \mu(A)$.

$M_T(\Omega)$ is nonempty, weak* compact, convex sitting in $N(\Omega)$ locally convex Hausdorff TVS.

An *extreme point* is a point in a convex set that canNOT be written as a non-trivial convex combination of two distinct points in the set; here, non-trivial means $c = ta + (1-t)b, a \neq b, 0 < t < 1$, i.e., $c \in int[a, b]$.

Let X be a vector space and $A \subset X$. The *closed convex hull* of A is the intersection of all closed convex sets containing A, equivalently the closure of the intersection of all convex sets containing A

In some sense, the closed convex hull of A is the set of all limits of convex combinations of elements of A. So it is the set of all limits of weighted averages of elements of A.

Krein-Milman Theorem: Let X be a locally convex Hausdorff Topological Vector Space. Let $K \subset X$ be a compact convex set. Then K is the closed convex hul of its extreme points.

Application to: $M_T(\Omega)$:

Defn: An MPT T is *ergodic* w.r.t. μ if for all $A \in \mathcal{A}$ whenever $T^{-1}(A) = A$, then $\mu(A)$ is 0 or 1.

Examples: Doubling map and irrational rotations.

Ergodic Theorem: An MPT T is ergodic iff for all $f \in L^1(\mu)$, for μ-a.e. $x \in \Omega$

$$\lim_{n \to \infty} (1/n) \sum_{i=0}^{n-1} f(T^i x) = \int f d\mu$$

"Time average = space average"

Theorem: Let T be continuous on a compact metric Ω. Then T is ergodic w.r.t $\mu \in M_T(\Omega)$ iff μ is an extreme point of $M_T(\Omega)$.

So, Krein-Milman Theorem implies:

— the existence of an ergodic invariant measure in $M_T(\Omega)$ for any continuous T.

— a decomposition of invariant measures as "averages" (limits of convex combinations) of ergodic invariant measures.

THE END