Lecture 35:
Recall:
– \( \Omega \) is a compact metric space
– \( M(\Omega) \) denotes the set of all Borel probability measures on \( \Omega \).
– \( M(\Omega) \) is contained in the unit ball of \( N(\Omega) \), the set of all finite signed Borel measures on \( \Omega \), which can be regarded as the unit ball of \( C(\Omega)^* \).
– \( M(\Omega) \) is weak* sequentially compact: every sequence in \( M(\Omega) \) has a convergent subsequence that converges to an element of \( M(\Omega) \).

– Weak* sequential convergence in \( M(\Omega) \): \( \mu_n \xrightarrow{wk^*} \mu \) iff for all \( f \in C(\Omega) \), \( \int f d\mu_n \to \int f d\mu \), because the maps \( \mu \mapsto \int f d\mu \) are the linear functionals on \( N(\Omega) \) that define the weak* topology.

Let \( T : \Omega \to \Omega \) be continuous (think of the doubling map or a rotation on the circle). Then \( T \) acts on \( M(\Omega) \):

\[
(\tilde{T}\mu)(A) = \mu(T^{-1}(A)).
\]

Defn: \( M_T(\Omega) := \{ \mu \in M(X) : \tilde{T}\mu = \mu \} \).

The elements of \( M_T(\Omega) \) are called \( T \)-invariant (Borel probability) measures; then \( T \) can be viewed as an MPT w.r.t. each \( T \)-invariant measure.

Example: For the doubling map, in addition to Lebesgue measure, \( \delta_0 \) is an invariant measure.

Example: For rotation by \( \pi/2 \), in addition to Lebesgue measure, the periodic point measure of period 4 starting at any point is an invariant measure.

Prop 1: Let \( T : \Omega \to \Omega \) be continuous and \( \mu \in M(\Omega) \). Then \( \mu \in M_T(\Omega) \) iff for all \( f \in C(\Omega) \), \( \int f \circ T d\mu = \int f d\mu \).
Lemma 1: Let $\mu, \nu \in M(X)$. Then $\int f \, d\mu = \int f \, d\nu$ for all $f \in C(\Omega)$ iff $\mu = \nu$.

Proof: Apply Approximation Lemma. □

Lemma 2: Let $T : \Omega \to \Omega$ be continuous and $\mu \in M(X)$. Then for all $f \in C(X)$ $\int f \, d\tilde{T}\mu = \int f \circ T \, d\mu$.

Proof: By defn of $\tilde{T}$, for all $A \in \mathcal{A}$

$$\int 1_A d\tilde{T}\mu = (\tilde{T}\mu)(A) = \mu(T^{-1}(A)) = \int 1_A \circ T \, d\mu$$

Thus for all simple functions $h$

$$\int h d\tilde{T}\mu = \int h \circ T \, d\mu$$

The same formula holds for all non-negative measurable functions $h$ by choosing an increasing sequence of simple functions converging pointwise to $h$.

But then the same formula holds for any continuous function by decomposing it into its positive and negative parts. □

Proof of Prop 1:

By defn, $\mu \in M_T(X)$ iff $\mu = \tilde{T}\mu$.

By Lemmas 1 and 2, $\mu = \tilde{T}\mu$ iff for all $f \in C(X)$ $\int f \, d\mu = \int f \, d\tilde{T}\mu = \int f \circ T \, d\mu$. □

Defn: For $x \in \Omega$, $\delta_x$ is the measure which assigns 1 to a set that contains $x$ and 0 otherwise.

Theorem (Krylov and Bolgoliubov): $M_T(\Omega) \neq \emptyset$

Proof: Fix $x \in \Omega$. Let

$$\mu_n = (1/n) \sum_{i=0}^{n-1} \tilde{T}^i \delta_x \in M(X)$$
Since $M(\Omega)$ is weak* sequentially compact, some subsequence $\mu_{n_j}$ converges weak* to some $\mu \in M(\Omega)$. We claim that $\mu \in M_T(\Omega)$.

By defn. of weak* convergence, for all $f \in C(\Omega)$, $\int f d\mu_{n_i} \to \int f d\mu$ and $\int f \circ T d\mu_{n_i} \to \int f \circ T d\mu$.

Let $f \in C(X)$. Then

$$|\int f \circ T d\mu - \int f d\mu| = \lim_{i \to \infty} |\int f \circ T d\mu_{n_i} - \int f d\mu_{n_i}|$$

$$= \lim_{i \to \infty} |(1/n_i) \sum_{j=0}^{n_i-1} (f \circ T^{j+1} - f \circ T^j) d\delta_x|$$

$$= \lim_{i \to \infty} |(1/n_i) \int (f \circ T^{n_i} - f) d\delta_x|$$

$$\leq \lim_{i \to \infty} \frac{2||f||_{\text{sup}}}{n_i} = 0$$

Thus, by Prop 1, $\mu \in M_T(\Omega)$. □

Prop: $M_T(\Omega)$ is a weak* compact subset of $M(\Omega)$.

Proof: By Prop 1,

$$M_T(\Omega) = \{\mu \in M(\Omega) : \forall f \in C(\Omega), \int f \circ T d\mu = \int f d\mu.\}$$

Let $\mu_n \in M_T(\Omega)$ and $\mu_n \overset{wk^*}{\to} \mu \in M(\Omega)$. Then for all $f \in C(\Omega)$, $f \circ T \in C(\Omega)$, and so

$$\int f \circ T d\mu = \lim_n \int f \circ T d\mu_n = \lim_n \int f d\mu_n = \int f d\mu$$

So $\mu \in M_T(\Omega)$.

So, $M_T(\Omega)$ is weak*-closed. Since $M(\Omega)$ is weak*-compact, $M_T(\Omega)$ is weak* compact. □

So, $M_T(\Omega)$ is a non-empty weak* compact, convex subset of a locally convex TVS.