Lecture 35:

Recall:

– Ω is a compact metric space

– $M(\Omega)$ denotes the set of all Borel probability measures on Ω.

– $M(\Omega)$ is contained in the unit ball of $N(\Omega)$, the set of all finite signed Borel measures on Ω, which can be regarded as the unit ball of $C(\Omega)^*$.

– $M(\Omega)$ is weak* sequentially compact: every sequence in $M(\Omega)$ has a convergent subsequence that converges to an element of $M(\Omega)$.

– Weak* sequential convergence in $M(\Omega)$: $\mu_n \overset{wk*}{\to} \mu$ iff for all $f \in C(\Omega)$, $\int f d\mu_n \to \int f d\mu$, because the maps $\mu \mapsto \int f d\mu$ are the linear functionals on $N(\Omega)$ that define the weak* topology.

Let $T : \Omega \to \Omega$ be continuous (think of the doubling map or a rotation on the circle). Then T acts on $M(\Omega)$:

$$(\tilde{T}\mu)(A) = \mu(T^{-1}(A).$$

Defn: $M_T(\Omega) := \{\mu \in M(X) : \tilde{T}\mu = \mu\}$.

The elements of $M_T(\Omega)$ are called *T-invariant (Borel probbility) measures*; then T can be viewed as an MPT w.r.t. each T-invariant measure.

Example: For the doubling map, in addition to Lebesgue measure, δ_0 is an invariant measure.

Example: For rotation by $\pi/2$, in addition to Lebesgue measure, the periodic point measure of period 4 starting at any point is an invariant measure.

Prop 1: Let $T : \Omega \to \Omega$ be continuous and $\mu \in M(\Omega)$. Then $\mu \in M_T(\Omega)$ iff for all $f \in C(\Omega)$, $\int f \circ T d\mu = \int f d\mu$.

Lemma 1: Let $\mu, \nu \in M(X)$. Then $\int f d\mu = \int f d\nu$ for all $f \in C(\Omega)$ iff $\mu = \nu$.

Proof: Apply Approximation Lemma. $\square$

Lemma 2: Let $T : \Omega \to \Omega$ be continuous and $\mu \in M(X)$. Then for all $f \in C(X)$ $\int f d\tilde{T}\mu = \int f \circ T d\mu$.

Proof: By defn of $\tilde{T}$, for all $A \in \mathcal{A}$

$$\int 1_A d\tilde{T}\mu = (\tilde{T}\mu)(A) = \mu(T^{-1}(A)) = \int 1_A \circ T d\mu$$

Thus for all simple functions h

$$\int h d\tilde{T}\mu = \int h \circ T d\mu$$

The same formula holds for all non-negative measurable functions h by choosing an increaeing sequence of simple functions converging pointwise to h.

But then the same formula holds for any continuous function by decompositing it into its positive and negative parts. $\square$

Proof of Prop 1:

By defn, $\mu \in M_T(X)$ iff $\mu = \tilde{T}\mu$.

By Lemmas 1 and 2, $\mu = \tilde{T}\mu$ iff for all $f \in C(X)$ $\int f d\mu = \int f d\tilde{T}\mu = \int f \circ T d\mu$. $\square$

Defn: For $x \in \Omega$, δ_x is the measure which assigns 1 to a set that contains x and 0 otherwise.

Theorem (Krylov and Bolgoliubov): $M_T(\Omega) \neq \emptyset$

Proof: Fix $x \in \Omega$. Let

$$\mu_n = (1/n) \sum_{i=0}^{n-1} \tilde{T}^i \delta_x \in M(X)$$

Since $M(\Omega)$ is weak* sequentially compact, some subsequence μ_{n_j} converges weak* to some $\mu \in M(\Omega)$. We claim that $\mu \in M_T(\Omega)$.

By defn. of weak* convergence, for all $f \in C(\Omega)$, $\int f d\mu_{n_i} \to \int f d\mu$ and $\int f \circ T d\mu_{n_i} \to \int f \circ T d\mu$.

Let $f \in C(X)$. Then

$$\begin{aligned}
|\int f \circ T d\mu - \int f d\mu| &= \lim_{i\to\infty} |\int f \circ T d\mu_{n_i} - \int f d\mu_{n_i}| \\
&= \lim_{i\to\infty} |(1/n_i) \int \sum_{j=0}^{n_i - 1} (f \circ T^{j+1} - f \circ T^j) d\delta_x| \\
&= \lim_{i\to\infty} |(1/n_i) \int (f \circ T^{n_i} - f) d\delta_x| \\
&\leq \lim_{i\to\infty} \frac{2||f||_{\text{sup}}}{n_i} = 0
\end{aligned}$$

Thus, by Prop 1, $\mu \in M_T(\Omega)$. □

Prop: $M_T(\Omega)$ is a weak* compact subset of $M(\Omega)$.

Proof: By Prop 1,

$$M_T(\Omega) = \{\mu \in M(\Omega) : \forall f \in C(\Omega), \int f \circ T d\mu = \int f d\mu.$$

Let $\mu_n \in M_T(\Omega)$ and $\mu_n \overset{wk*}{\to} \mu \in M(\Omega)$. Then for all $f \in C(\Omega)$, $f \circ T \in C(\Omega)$, and so

$$\int f \circ T d\mu = \lim_n \int f \circ T d\mu_n = \lim_n \int f d\mu_n = \int f d\mu$$

So $\mu \in M_T(\Omega)$.

So, $M_T(\Omega)$ is weak*-closed. Since $M(\Omega)$ is weak*-compact, $M_T(\Omega)$ is weak* compact. □

So, $M_T(\Omega)$ is a non-empty weak* compact, convex subset of a locally convex TVS.