Lecture 29:

Recall Props 1,2,3.

Remark: For a topological space $(X, \mathcal{T})$ and $A \subseteq X$, let $\overline{A}^{\mathcal{T}}$ denote the closure of A w.r.t. $\mathcal{T}$.

If $\mathcal{T}_1 \subseteq \mathcal{T}_2$, then $\overline{A}^{\mathcal{T}_1} \supseteq \overline{A}^{\mathcal{T}_2}$ because in a weaker topology because there are fewer open sets and therefore fewer closed sets. Here is an extreme example.

Recall that in the norm topology the unit sphere is closed.

Theorem: The weak closure (i.e., the closure in the weak topology) of the unit sphere in an infinite-dimensional NVS is the closed unit ball.

Proof: Recall that $\overline{B}_1(0)$ denotes the closed unit ball and S^1 denotes the unit sphere.

First, we claim that

$$\overline{B}_1(0) = \bigcap_{\{f \in X^* : ||f|| = 1\}} \{x : |f(x)| \leq 1\}$$

Proof of Claim: If $||f|| = 1$, then $|f(x)| \leq ||x||$. So, if $||x|| \leq 1$, then $|f(x)| \leq 1$ and so LHS $\subseteq$ RHS.

By Hahn-Banach corollary (Theorem 5.8b), for all $x \neq 0$, there exists $f \in X^*$ s.t. $||f|| = 1$ and $|f(x)| = ||x||$. So, if $x \notin \overline{B}_1(0)$, then $||x|| > 1$, and so $|f(x)| > 1$ and so $x \notin$ RHS. Thus LHS $\supseteq$ RHS$\square$

Now, the RHS is the intersection of weakly closed sets and is thus weakly closed. Thus, $\overline{B}_1(0)$ is a weakly closed set that contains S^1. Thus, it contains the weak closure of S^1.

By Prop 2, it remains to show that for every $x_0 \in \overline{B}_1(0)$, every nbhd U of x_0 intersects S^1. We may assume that U belongs to the nbhd. base (Prop 1) of x_0, i.e.,

$$U = \cap_{i=1}^n U_{f_i, x_0, \epsilon_i} = \bigcap_{i=1}^n \{x : |f_i(x - x_0)| < \epsilon_i\}$$

for some choice of $f_1, \ldots f_n \in X^*$ and $\epsilon_i > 0$.

By Prop 3, $\cap_{i=1}^n ker(f_i) \neq 0$.

Let $y \neq 0$ s.t. $y \in \cap_{i=1}^n ker(f_i)$ and let L be the (one-dimensional) linear span of y. Then $L + x_0$ is contained in U and in particular is a line which intersects the unit ball. Intuitively, such a line must intersect S^1 (and so U would intersect S^1). Here, is a precise proof:

First observe that the continuous function $f(\lambda) := ||\lambda y + x_0||$ satsifies $f(0) = ||x_0|| \leq 1$ and $\lim_{\lambda \to \infty} f(\lambda) = \infty$, the latter since

$$f(\lambda) \geq \lambda ||y|| - ||x_0||$$

It follows from the internmediate value theorem that for some λ_0, $f(\lambda_0) = 1$ and so $\lambda_0 y + x_0 \in U \cap S^1$. Thus, by Prop 2, x_0 belongs to the weak closure of S. $\square$

Cor: For an infinite-dimensional NVS, the weak topology is strictly weaker than the norm topology.

Fact: In finite-dimensional case: weak topology = norm topology.

Proof: The norm topology on $\mathbb{R}^n$ is the Euclidean topology.

The weak topology is the product topology of n copies of $\mathbb{R}$, because the coordinate projections form a basis of the BLFs.

From HW4 #7d, the Euclidean topology on $\mathbb{R}^n$ is the same as the product topology of n copies of $\mathbb{R}$. $\square$

For our next result, we will need the following

Lemma: Let W be a (norm-) closed subspace of an NVS X. Let $y \in X \setminus W$. Then there is an $f \in X^*$ s.t. $f(y) > 0$ and $f|_W = 0$.

In other words, a BLF can separate a point, not on W, from W.

Proof: Apply Hahn-Banach Theorem.

Let U be the subspace generated by y and W, i.e.,

$$U = \{\lambda y + w : w \in W\}$$

Define f on U by $f(\lambda y + w) = \lambda \delta$ where

$$\delta = d(y, W) > 0 \text{ since W is norm-closed}$$

Then f is linear, $f(y) > 0$ and $f|_W = 0$.

Now by HB it suffices to show that f is a BLF on U. For this observe that if $\lambda \neq 0$, then

$$||\lambda y + w|| = |\lambda| \; ||y + w/\lambda|| \geq |\lambda|\delta = |f(\lambda y + w)|$$

so in fact $||f|_U|| \leq 1$. $\square$

Streamlined proof (Taek): Since W is closed, X/W is a NVS. And the equivalence class $[y] \neq 0$. Apply HB to get a $g \in (X/W)^*$ s.t. $g([y]) > 0$. The compositoin $f = g \circ q$, where $q : X \to X/W$ is the quotient mapping, is a BLF that satisfies $f(y) > 0$ and $f|_W = 0$.

Theorem; A subspace of an NVS is weakly closed iff it is norm-closed.

Proof: next time.