Lecture 28:

Let $(X, \mathcal{T})$ be a topological space.

By a *neighbourhood* of a point $x \in X$ we mean an open set containing x.

Defn: A sequence x_i *converges* to x if for each nbhd U of x, there exists N s.t. for all $i \geq N$, $x_i \in U$.

Bases and Neighbourhood Bases:

Defn: A *base* for a topology $\mathcal{T}$ is a sub-collection $\mathcal{B} \subset \mathcal{T}$ such that $\mathcal{T}$ is the collection of all unions of elements of $\mathcal{B}$.

Usefulness of bases:

Given topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ with base $\mathcal{B}$ for $\mathcal{S}$, $f : X \to Y$ is continuous iff $f^{-1}(B) \in \mathcal{T}$ for each $B \in \mathcal{B}$.

Every topological space has a base, namely $\mathcal{B} = \mathcal{T}$. This is not a very useful base; the smaller the base, the better.

Example: For the topology of a metric space, the collection $\mathcal{B}$ of all open balls is a base.

Read criterion for base: Prop. 4.3.

Defn: A *neighbourhood base* for $\mathcal{T}$ at a point $x_0 \in X$ is a collection $\mathcal{B}_{x_0} \subset \mathcal{T}$ s.t. each $B \in \mathcal{B}_{x_0}$ contains x_0 and for all $U \in \mathcal{T}$ s.t. $x_0 \in U$, there exists some $B \in \mathcal{B}_{x_0}$ s.t.

$$x_0 \in B \subset U$$

Example: For the topology of a metric space, the collection $\mathcal{B}_{x_0}$ of open balls centered at x_0 is a nbhd base at x_0.

Remark: It is easy to see that a collection $\mathcal{B} \subset \mathcal{T}$ is a base for $\mathcal{T}$ iff it is a union of nhbhd. bases at all $x_0 \in X$.

Proof:

If $\mathcal{B}$ is a base, then each $U \in \mathcal{T}$ is a union of elements of $\mathcal{B}$ and so $\{V \in \mathcal{B} : x \in V\}$ is a nbhd. base at x.

Conversely, if $\mathcal{B}$ is a union of nbhd bases, $U \in \mathcal{T}$ and $x \in U$, then there exists $V_x \in \mathcal{B}$ s.t.

$$x \in V_x \subset U$$

and so $U = \cup_x V_x$. So, every $U \in \mathcal{T}$ is a union of elements of $\mathcal{B}$. $\square$

Given topologies $\mathcal{T}_1 \subset \mathcal{T}_2$, we say that $\mathcal{T}_1$ is *coarser* or *weaker* than $\mathcal{T}_2$ and that $\mathcal{T}_2$ is *finer* or *stronger* than $\mathcal{T}_2$.

Defn: The *weak topology* of an NVS X is the weakest topology on X s.t. each $f \in X^*$ is continuous, meaning the intersection of all topologies on X such that every $f \in X^*$ is continuous.

Prop 1:

a. The weak topology is the topology generated by sets of the form

$$U_{f,x_0,\epsilon} = \{x \in X : |f(x - x_0)| < \epsilon\}, \quad f \in X^*, x_0 \in X, \epsilon > 0 \quad (1)$$

and so is the collection of all unions of finite intersections of sets of this form.

b. A base for the weak topology is the set of all finite intersections of the form (1).

c. A neighbourhood base for the weak topology at x^* is the set of all finite intersections of the form (1) with $x_0 = x^*$.

Proof:

a. and b. are clear.

c. If $x^* \in X$, and $U \in \mathcal{T}$, then there exists some $\cap_{i=1}^n U_{f_i,x_i,\epsilon_i} \in \mathcal{T}$ s.t.

$$x^* \in \cap_{i=1}^n U_{f_i,x_i,\epsilon_i} \subset U$$

But then, defining $\delta_i := \epsilon_i - |f_i(x^* - x_i)|$, we have, by the triangle inequality, we have

$$|f_i(x - x^*)| < \delta_i \Rightarrow |f_i(x - x_i)| < \epsilon_i$$

$$x^* \in \cap_{i=1}^n U_{f_i, x^*, \delta_i} \subset \cap_{i=1}^n U_{f_i, x_i, \epsilon_i} \subset U$$

Remarks:

1. The weak topology is weaker than the norm topology because all elements of X^* are continuous w.r.t. the norm topology. The weak topology turns out to coincide with the norm topology in and only in the finite dimensional case.

2. Using the framework of semi-norm topologies in Folland (section 5.4), one can show that the weak topology on an NVS is a TVS.

Defn: In a topological space, the *closure* of a set A is the smallest closed set containing A, i.e., the intersection of all closed sets containing A.

Prop 2: (HW5): In a topological space, the closure of A is the set of all $x \in X$ s.t. each nbhd. U of x intersects A.

A bit of linear algebra:

Let X be a vector space and W be a subspace. Recall that the quotient space X/W is the set of equivalence classes with equivalence relation: $x \sim y$ if there exists $w \in W$ s.t. $x = y + w$. The quotient space inherits a vector space structure (HW4#4).

The *codimension* of W in X is defined to be:

$$\mathrm{codim}(W) := \dim(X/W).$$

Let $W_1, \ldots, W_n$ be subspaces of X. The map

$$X/(\cap_{i=1}^n W_i) \to (X/W_1) \oplus \ldots \oplus (X/W_n), \quad [x]_{\cap_{i=1}^n W_i} \mapsto [x]_{W_1} \oplus \ldots \oplus [x]_{W_n}$$

is a (well-defined) linear injection because the kernel of this map is 0.

Thus,

$$\text{codim}(\cap_{i=1}^{n} W_i) \leq \sum_{i-1}^{n} \text{codim}(W_i)$$

In particular, if each W_i has finite codimension, then so does the intersection.

Now, suppose that X is a vector space and $\phi : X \to K$ is a linear functional.

If $\phi \equiv 0$, then $X/(\ker\phi) = 0$ and so $codim(\ker\phi) = 0$.

If $\phi \not\equiv 0$, then by the first isomorphism theorem of groups $X/(\ker\phi) \approx K$ and so $\text{codim}(\ker\phi) = 1$.

Prop 3: If X is infinite-dimensional and $\phi_1, \ldots, \phi_n$ are linear functionals, then $\cap ker\phi_i \neq (0)$.

Proof: $\text{codim}(\cap_{i=1}^{n}\ker\phi_i) \leq \sum_{i=1}^{n} \text{codim}(\ker\phi_i) \leq n$.

If $\cap_{i=1}^{n}\ker\phi_i = (0)$, then $X/\cap_{i-1}^{n}\ker\phi_i = X$ and so $\text{codim}(\cap_{i=1}^{n}\ker\phi_i) = \infty$, a contradiction. $\square$