Lecture 27:

Recall:

Defn: Let X and Y be topological spaces. A mapping $f : X \to Y$ is *closed* if its graph $G := \{(x, T(x)) : x \in X\}$ is closed in $X \times Y$.

Closed graph theorem: Let T be a linear map from a Banach space X into a Banach space Y. Let be the graph of T. Then T is continuous iff T is closed.

Recall: "only if:" $\Rightarrow$ (continuous implies closed) was the easy part and does not require X or Y to be Banach.

"if" $\Leftarrow$ (closed implies continuous): requires X and Y to be Banach.

Continuous is the a priori stronger property
$\therefore$ the 1st direction is more useful

Uniform Bddness Principle (Banach-Steinhaus Theorem): Let X be a Banach space and Y a NVS and A a collection of BLTs from X into Y. If for all $x \in X$, $\sup_{T \in A} ||Tx|| < \infty$, then $\sup_{T \in A} ||T|| < \infty$.

Can be proven by Open Mapping or Closed Graph Theorem.

Proof: Proof directly by BCT.

Let

$$E_n = \{x \in X : ||Tx|| \leq n, \forall T \in A\}$$

Then

$$X = \cup_n E_n.$$

Also, since each $T \in A$ is continuous, $\{x \in X : ||Tx|| \leq n\}$ is closed and thus E_n is the intersection of closed sets and thus is closed.

Since X is complete and each E_n is closed, for some n, $int(E_n) \neq \emptyset$. So for some $x_0 \in E_n$ and $r > 0$, $\overline{B_r(x_0)} \subset E_n$.

Thus, if $x \in \overline{B_r(0)}$, then $x + x_0 \in E_n$ and so for all $T \in A$,

$$||Tx|| \leq ||Tx_0|| + ||T(x_0 + x)|| \leq 2n$$

Thus, $||T|| \leq 2n/r$. □

Application: Let X be a Banach space and Y a NVS. Let T_n be a sequence of BLTs from X into Y s.t. for each $x \in X$ $\lim_{n\to\infty} T_n x$ exists. Define

$$Tx := \lim_{n\to\infty} T_n x$$

$A = \{T_n\}$

Then T is a BLT.

Proof: Since each T_n is linear, it is easy to see that T is linear.

In a NVS for any convergent sequence y_n, $\sup_n ||y_n|| < \infty$. Thus, for each x, $\sup_n ||T_n x|| < \infty$. Thus, by uniform boundedness $\sup_n ||T_n|| < \infty$, say $\sup_n ||T_n|| \leq M$. Thus, for each $x \in X$,

$$||Tx|| = \lim_{n\to\infty} ||T_n x|| \leq \limsup_{n\to\infty} ||T_n||\ ||x|| \leq M||x||.$$

So, T is a BLT. □

Another application: A subset S of a NVS X is bounded if for all $f \in X^*$, $\sup_{x\in S} |f(x)| < \infty$.

Proof: Recall that X^* is a Banach space. Let $\hat{S}$ be the image of S via the canonical embedding $X \to X^{**}$, $x \mapsto \hat{x}$. Apply the UBP to $\mathcal{A} = \hat{S}$. The assumption becomes for all $f \in X^*$,

$$\sup_{\hat{x}\in\hat{S}} |\hat{x}(f)| < \infty$$

Apply UBP to $A = \hat{S}$ Banach X^* $Y = K$

The conclusion of UBP is that

$$\sup_{\hat{x}\in\hat{S}} ||\hat{x}|| < \infty$$

But since $||\hat{x}|| = ||x||$, S is bounded. □

Defn: A *topological vector space (TVS)* is a vector space, together with a topology, such that vector addition and scalar mulitplication are continuous, i.e., the maps

$$X \times X \to X, (x, y) \mapsto x + y \text{ and } \quad K \times X \to X, (\lambda, x) \mapsto \lambda x$$

are continuous w.r.t. the product topology on the domains.

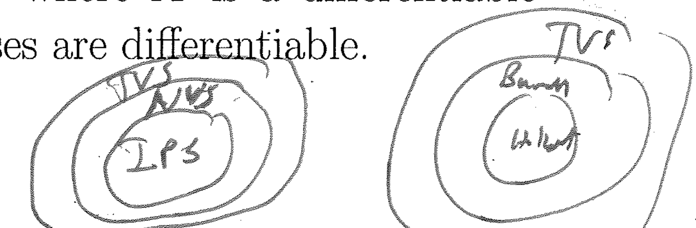
You can consider other combinations of geometric/algebraic structures. For instance,

Defn: A *topological group* is a group, together with a topology, such that multiplication and inverses are continuous, i.e., the maps

$$X \times X \to X, (x, y) \mapsto xy \text{ and } X \to X, x \mapsto x^{-1}$$

are continuous.

A Lie group is a topological group where X is a differentiable manifold and multiplication and inverses are differentiable.



Prop: Any NVS is a TVS.

Proof:

As mentioned earlier, for NVS's X and Y, the topology of the product norm, $||(x, y)|| = ||x|| + ||y||$, is the same as the product topology of $X \times Y$, and convergence $(x_n, y_n) \to (x, y)$ is the same as $x_n \to x$ and $y_n \to y$.

Continuity of vector addition:

$x_n \to x$ & $y_n \to y$ $\Rightarrow$ $x_n + y_n \to x + y$

$$||(x, y)|| := ||x|| + ||y||$$

$$||x_n + y_n - (x + y)|| \leq ||x_n - x|| + ||y_n - y||$$

So, as $x_n \to x$ and $y_n \to y$, $x_n + y_n \to x + y$.

Continuity of scalar multiplication:

$x_n \to x$ & $\lambda_n \to \lambda$ $\Rightarrow$ $\lambda_n x_n \to \lambda x$.

$$||(\lambda, x)|| := |\lambda| + ||x||$$

$$||\lambda_n x_n - \lambda x|| = ||(\lambda_n x_n - \lambda_n x) + (\lambda_n x - \lambda x)|| \leq |\lambda_n|\ ||x_n - x|| + |\lambda_n - \lambda|\ ||x||$$
$$\leq 2|\lambda|\ ||x_n - x|| + |\lambda_n - \lambda|\ ||x||$$

the latter for sufficiently large n.

So, as $x_n \to x$ and $\lambda_n \to \lambda$, $\lambda_n x_n \to \lambda x$.