Lecture 26:

Recall:

Open Mapping Theorem: Every continuous linear map (BLT) *from a Banach space X onto a Banach space Y* is an open mapping, i.e., the image of an open set is open.

Remark: In proof of Open Mapping theorem, completeness of Y and surjectivity were used in Step 1. Completeness of X is used in Step 3.

Remark: The open mapping theorem is false if we replace "open" by closed, meaning that the image of each closed set is closed.

– Example: The hyperbola $U := \{(x, y) : xy = 1,\ x > 0\}$ in $\mathbb{R}^2$, with T being the projection to $\mathbb{R}$ (on either the x-axis or y-axis). Here, U is closed but its projection is not closed.

Corollary: Any bijective continuous linear map from a Banach space onto a Banach space is a homeomorphic isomorphism.

Proof: By the open mapping theorem, the inverse of the map is continuous. So, it is a homeomorphism. We showed long ago that the inverse of a linear bijection is linear and so it is a linear isomorphism. $\square$

Application: Let X be a vector space with two norms, $||\cdot||_1, ||\cdot||_2$, with respect to each of which X is a Banach space and s.t. there is a constant $C > 0$ s.t. for all $x \in X$

$$||x||_1 \leq C||x||_2$$

Then the two norms are equivalent.

Proof: The identity map $(X, ||\cdot||_2) \to (X, ||\cdot||_1)$ is a bijective continuous linear map and therefore a homeomorphism. Thus, the topologies of the two NVS are the same. But then by HW1 #5b, the two norms are equivalent. $\square$

Example: The application is false without the Banach assumption. For the NVS c_c we can consider two norms: $||\cdot||_1 =$ sup norm and $||\cdot||_2 = \ell^1$. Then $||x||_{\text{sup}} \leq ||x||_{\ell^1}$ but for x^n defined by $x_i^n = 1$ for $i \leq n$ and $x_i^n = 0$ for $i > n$, we have $||x^n||_{\text{sup}} = 1$, $||x^n||_{\ell^1} = n$ and so the two norms are not equivalent. Here, the problem is that these NVS are not Banach.

Defn: Let $(X, ||\cdot||_X)$ and $(Y, ||\cdot||_Y)$ be NVS. The product NVS is $(X \times Y, ||\cdot||)$ with norm $||(x,y)|| = ||x||_X + ||y||_Y$.

Remarks:

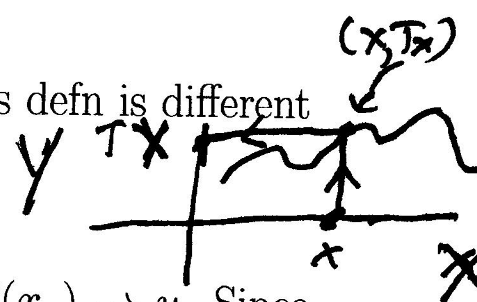
1. in the topology of the product norm, $(X \times Y, ||\cdot||)$, $(x_n, y_n) \to (x,y)$ iff $x_n \to x$ and $y_n \to y$.

2. the product of Banach spaces is a Banach space

3. Exercise: the product topology of the topologies of $(X, ||\cdot||_X)$ and $(Y, ||\cdot||_Y)$ is the same as the topology of the product norm.

Corollary (The closed graph theorem): Let T be a linear map from a Banach space X into a Banach space Y. Let $G := \{(x, T(x)) : x \in X\}$ be the graph of T. Then T is continuous iff G is closed in $X \times Y$.

Defn: A map is *closed* if its graph is closed (this defn is different from the defn given in HW4 #6).



Proof: "Only If:"

Let $(x_n, T(x_n)) \to (x,y)$. Then $x_n \to x$ and $T(x_n) \to y$. Since T is continuous $T(x_n) \to T(x)$. So, $(x,y) = (x, T(x)) \in G$.

"If:"

Let $\pi_X : X \times Y \to X$ be projection onto X and $\pi_Y : X \times Y \to Y$ be projection onto Y. Then $T = \pi_Y \circ (\pi_X|_G)^{-1}$. Since G is a closed subset of the Banach space $X \times Y$, it is a Banach space. So, $\pi_X|G$ is continuous and bijective from the Banach space G onto the Banach space X . By the open mapping theorem, $\pi_X|G$ is a homeomorphism and so $(\pi_X|_G)^{-1}$ is continuous. So, T is the composition of two continuous maps and is thus continuous. $\square$.

Consider the following statements:

1. x_n converges to some element x
2. Tx_n converges to some element y
3. $Tx = y$

To shows that T is continuous, one must show that 1 implies 2 and 3.

The closed graph theorem shows that for a linear map from one Banach space to another, it is sufficient to show that 1 and 2 imply 3, which is easier.

Example: Let $C([0,1])$ denote the set of continuous functions and and $C^1([0,1])$ the set of ctsly diffble functions on $[0,1]$, both NVS with the sup norm. Let $T : C^1([0,1]) \to C([0,1])$ be the derivatiive map. Then the graph of T is closed but T is not continuous.

Proof: T is a linear functional but is not bounded, and thus not continuous, because the functions $t^n \in C^1([0,1])$ have norm 1 and

$$||T(t^n)|| = ||(t^n)'|| = ||nt^{n-1}|| = n \to \infty.$$

To show that the graph of T is closed, we must show that if $f_n \in C^1([0,1])$ and both f_n converges uniformly to some $f \in C([0,1])$ and f_n' converges uniformly to some $g \in C([0,1])$, then $g = f'$. But this is a result in third year analysis. $\square$

The problem here is that the domain $C^1([0,1])$ is not complete.