Lecture 26:

Recall:

Open Mapping Theorem: Every continuous linear map (BLT) from a Banach space $X$ onto a Banach space $Y$ is an open mapping, i.e., the image of an open set is open.

Remark: In proof of Open Mapping theorem, completeness of $Y$ and surjectivity were used in Step 1. Completeness of $X$ is used in Step 3.

Remark: The open mapping theorem is false if we replace “open” by closed, meaning that the image of each closed set is closed.

- Example: The hyperbola $U := \{(x, y) : xy = 1, \ x > 0\}$ in $\mathbb{R}^2$, with $T$ being the projection to $\mathbb{R}$ (on either the $x$-axis or $y$-axis). Here, $U$ is closed but its projection is not closed.

Corollary: Any bijective continuous linear map from a Banach space onto a Banach space is a homeomorphic isomorphism.

Proof: By the open mapping theorem, the inverse of the map is continuous. So, it is a homeomorphism. We showed long ago that the inverse of a linear bijection is linear and so it is a linear isomorphism. 

Application: Let $X$ be a vector space with two norms, $\| \cdot \|_1, \| \cdot \|_2$, with respect to each of which $X$ is a Banach space and s.t. there is a constant $C > 0$ s.t. for all $x \in X$

$$\|x\|_1 \leq C\|x\|_2$$

Then the two norms are equivalent.

Proof: The identity map $(X, \| \cdot \|_2) \rightarrow (X, \| \cdot \|_1)$ is a bijective continuous linear map and therefore a homeomorphism. Thus, the topologies of the two NVS are the same. But then by HW1 #5b, the two norms are equivalent. \(\Box\)
Example: The application is false without the Banach assumption. For the NVS $c_c$ we can consider two norms: $|| \cdot ||_1 = \sup$ norm and $|| \cdot ||_2 = \ell^1$. Then $||x||_{\sup} \leq ||x||_{\ell^1}$ but for $x^n$ defined by $x^n_i = 1$ for $i \leq n$ and $x^n_i = 0$ for $i > n$, we have $||x^n||_{\sup} = 1$, $||x^n||_{\ell^1} = n$ and so the two norms are not equivalent. Here, the problem is that these NVS are not Banach.

Defn: Let $(X, || \cdot ||_X)$ and $(Y, || \cdot ||_Y)$ be NVS. The product NVS is $(X \times Y, || \cdot ||)$ with norm $||(x, y)|| = ||x||_X + ||y||_Y$.

Remarks:

1. in the topology of the product norm, $(X \times Y, || \cdot ||)$, $(x_n, y_n) \to (x, y)$ iff $x_n \to x$ and $y_n \to y$.

2. the product of Banach spaces is a Banach space

3. Exercise: the product topology of the topologies of $(X, || \cdot ||_X)$ and $(Y, || \cdot ||_Y)$ is the same as the topology of the product norm.

Corollary (The closed graph theorem): Let $T$ be a linear map from a Banach space $X$ into a Banach space $Y$. Let $G := \{(x, T(x)) : x \in X\}$ be the graph of $T$. Then $T$ is continuous iff $G$ is closed in $X \times Y$.

Defn: A map is closed if its graph is closed (this defn is different from the defn given in HW4 #6).

Proof: “Only If:”

Let $(x_n, T(x_n)) \to (x, y)$. Then $x_n \to x$ and $T(x_n) \to y$. Since $T$ is continuous $T(x_n) \to T(x)$. So, $(x, y) = (x, T(x)) \in G$.

“If:”

Let $\pi_X : X \times Y \to X$ be projection onto $X$ and $\pi_Y : X \times Y \to Y$ be projection onto $Y$. Then $T = \pi_Y \circ (\pi_X|_G)^{-1}$. Since $G$ is a closed subset of the Banach space $X \times Y$, it is a Banach space. So, $\pi_X|_G$ is continuous and bijective from the Banach space $G$ onto the Banach
space $X$. By the open mapping theorem, $\pi_X|G$ is a homeomorphism and so $(\pi_X|G)^{-1}$ is continuous. So, $T$ is the composition of two continuous maps and is thus continuous. \(\square\).

Consider the following statements:

1. $x_n$ converges to some element $x$
2. $Tx_n$ converges to some element $y$
3. $Tx = y$

To shows that $T$ is continuous, one must show that 1 implies 2 and 3.

The closed graph theorem shows that for a linear map from one Banach space to another, it is sufficient to show that 1 and 2 imply 3, which is easier.

Example: Let $C([0, 1])$ denote the set of continuous functions and and $C^1([0, 1])$ the set of ctsly differentiable functions on $[0, 1]$, both NVS with the sup norm. Let $T : C^1([0, 1]) \rightarrow C([0, 1])$ be the derivative map. Then the graph of $T$ is closed but $T$ is not continuous.

Proof: $T$ is a linear functional but is not bounded, and thus not continuous, because the functions $t^n \in C^1([0, 1])$ have norm 1 and

$$||T(t^n)|| = ||(t^n)'|| = ||nt^{n-1}|| = n \rightarrow \infty.$$  

To show that the graph of $T$ is closed, we must show that if $f_n \in C^1([0, 1])$ and both $f_n$ converges uniformly to some $f \in C([0, 1])$ and $f_n'$ converges uniformly to some $g \in C([0, 1])$, then $g = f'$. But this is a result in third year analysis. \(\square\)

The problem here is that the domain $C^1([0, 1])$ is not complete.