Lecture 25:

Open Mapping Theorem: Every continuous linear map (BLT) *from a Banach space X onto a Banach space Y* is an open mapping, i.e., the image of ~~an~~ each open set is open.

First Step 4

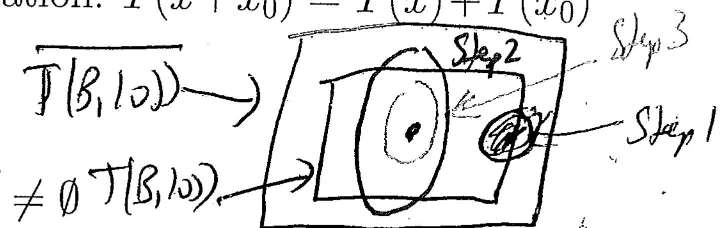
Note that for a BLT, the inverse image of any open set is open.

Proof: based on facts:

– T commutes with dilation: $T(\lambda x) = \lambda T(x)$ (since T is linear)

– T "semi-commutes" with translation: $T(x + x_0) = T(x) + T(x_0)$ (since T is linear)

Outline:



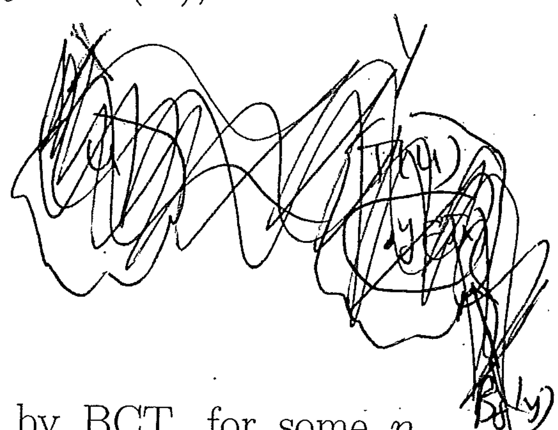
Step 1: Show that $int\overline{(T(B_1(0)))} \neq \emptyset$

Step 2: Show that $\overline{(T(B_1(0)))}$ contains an open ball centered at 0.

Step 3: Show that $T(B_1(0)))$ contains an open ball centered at 0

Step 4: Show for all open $U \subset X$ and $y \in T(U)$, there exists $\delta > 0$ s.t.

$$B_\delta(y) \subset T(U).$$

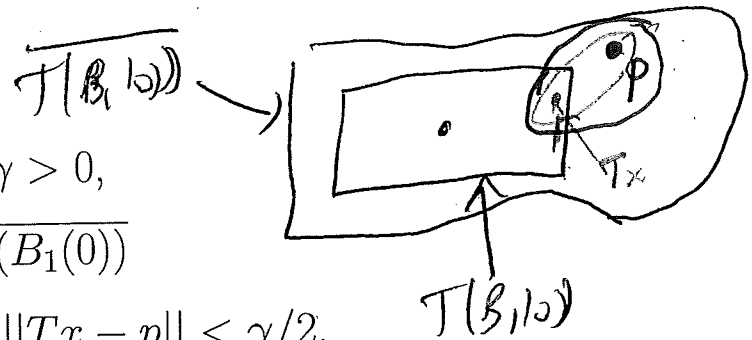


Step 1: Show that $int\overline{(T(B_1(0)))} \neq \emptyset$
Since T is onto,

$$Y = \cup_{n=1}^{\infty}(T(B_n(0)))$$

Thus since Y is a complete metric space, by BCT, for some n, $int(\overline{(T(B_n(0)))}) \neq \emptyset$. Then by dilation $int(\overline{(T(B_1(0)))}) \neq \emptyset$.

Step 2: Show that $\overline{(T(B_1(0)))}$ contains an open ball centered at 0.



By Step 1, for some point p and $\gamma > 0$,

$$B_\gamma(p) \subset \overline{T(B_1(0))}$$

There exists some $x \in B_1(0)$ s.t. $||Tx - p|| < \gamma/2$.

By the triangle inequality, $B_{\gamma/2}(Tx) \subset B_\gamma(p) \subset \overline{T(B_1(0))}$.

Thus, $B_{\gamma/2}(0) \subset \overline{T(B_1(0))} - Tx = \overline{T(B_1(0) - x)} \subset \overline{T(B_2(0))}$.

Dividing by 2, we find that

$$B_{\gamma/4}(0) \subset \overline{T(B_1(0))}$$

Step 3: Show that $T(B_1(0)))$ contains an open ball centered at 0.

Since by Step 2, for some $\gamma > 0$, $B_\gamma(0) \subset \overline{(T(B_1(0)))}$, by dilation it suffices to show that

$$\overline{(T(B_1(0)))} \subset T(B_2(0)).$$

Let $y \in \overline{(T(B_1(0)))}$. We will show that $y \in T(B_2(0))$.

Let $x_1 \in B_1(0)$ s.t. $||y - Tx_1|| < \gamma/2$ and so by dilation

$$y - Tx_1 \in B_{\gamma/2}(0) \subset \overline{(T(B_{1/2}(0)))}$$

Let $x_2 \in B_{1/2}(0)$ s.t. $||y - Tx_1 - Tx_2|| < \gamma/4$ and so

$$y - Tx_1 - Tx_2 \in B_{\gamma/4}(0) \subset \overline{(T(B_{1/4}(0)))}$$

Inductively choose $x_n \in B_{1/2^{n-1}}(0)$ s.t. $||y - \sum_{i=1}^n Tx_i|| < \gamma/2^n$ and so

$$y - \sum_{i=1}^{n} Tx_i \in B_{\gamma/2^n}(0) \subset \overline{(T(B_{1/2^n}(0)))}$$

Since X is complete and $\sum_{i=1}^\infty ||x_i|| < 2$, $\sum_{i=1}^\infty x_i$ converges to some $x \in B_2(0)$. And since $y - \sum_{i=1}^n Tx_i \to 0$ and T is continuous,

$$y = Tx$$

$\lim_{n\to\infty} \sum_{i=1}^{n} Tx_i =$ [illegible]

[illegible] $\sum x) = T(x)$

Thus, $y \in T(B_2(0))$.

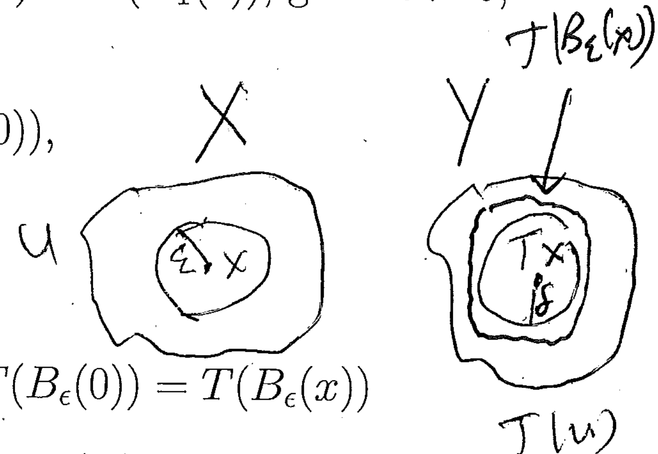
Step 4: Show for all open $U \subset X$ and $y \in T(U)$, there exists $\delta > 0$ s.t.

$$B_\delta(y) \subset T(U),$$

so $T(U)$ is open.

Since by Step 3, for some $\gamma > 0$, $B_\gamma(0) \subset T(B_1(0))$, given $\epsilon > 0$, there exists $\delta > 0$ s.t.

$$B_\delta(0) \subset T(B_\epsilon(0)),$$



Namely, by dilation, $\delta := \gamma\epsilon$.

Translate by adding Tx,

$$B_\delta(Tx) = Tx + B_\delta(0) \subset Tx + T(B_\epsilon(0)) = T(B_\epsilon(x))$$

Now, given an open U and $y = Tx \in T(U)$, there exists $\epsilon > 0$ s.t. $B_\epsilon(x) \subset U$ and so

$$B_\delta(y) = B_\delta(Tx) \subset T(B_\epsilon(x)) \subset T(U) \quad \square$$