Lecture 24:
Recall:

Defn: A subset $A$ of a topological space is *nowhere dense* if $\text{int}(\overline{A}) = \emptyset$, equivalently if $\overline{A}$ contains no open set.

Corollary (of Baire Category Theorem): A complete metric space cannot be written as a countable union of nowhere dense sets.

Theorem: A Hamel basis for a Banach space must be either finite or uncountable.

Proof: By contradiction. Suppose that a Banach space $X$ had a countably infinite Hamel basis $B := \{v_1, v_2, \ldots\}$.

Write

$$X = \bigcup X_n$$

where each $X_n := \text{span}(v_1, \ldots, v_n)$.

We will show:

a. each $X_n$ is closed in $X$

b. each $\text{int}(X_n) = \emptyset$,

contradicting BCT.

a. Follows from finite dimensionality of $X_n$: complete and therefore closed in $X$.

b. Let $x \in X_n$ and $\epsilon > 0$. Then $y := x + \epsilon v_{n+1} \notin X_n$, but $y \in B_{2\epsilon\|v_{n+1}\|}(x)$ and so $B_{2\epsilon\|v_{n+1}\|}(x) \nsubseteq X_n$ and so $\text{int}(X_n) = \emptyset$ $\square$

Note that the NVS, $c_c$ which is not complete, has a countable basis (the standard basis vectors, $e_i$ which is 1 in position $i$ and 0 elsewhere).
But the standard basis cannot be a Hamel basis for the Banach space $c_0$ because it can only generate sequences with finitely many non-zeros.

A more useful basis:

**Defn:** A *Schauder basis* for a NVS $X$ (in particular Banach $X$) is a countable set $\{v_n\}$ s.t. every $x \in X$ can be *expressed uniquely* as

$$x = \sum_n a_n v_n$$

meaning $\|x - \sum_{n=1}^{N} a_n v_n\| \to 0$.

Note that because of uniqueness any Schauder basis is linearly independent.

Clearly, any linear transformation on a vector space is determined by its values on a Hamel basis.

**Claim:** any continuous linear transformation (BLT) on a NVS is determined by its values on a Schauder basis.

**Proof:** Let $x = \sum_n a_n v_n \in X$, i.e., $\|x - \sum_{n=1}^{N} a_n v_n\| \to 0$.

Then by continuity of the norm, $\|Tx - \sum_{n=1}^{N} a_n Tv_n\| \to 0$ and so $Tx = \sum_n a_n Tv_n \in X$. $\square$

So, from the point of view of Banach spaces, Schauder bases are much more useful than Hamel bases.

Which Banach spaces have Schauder bases?

**Example:** any *separable* Hilbert space has a Schauder basis.

Recall that any separable Hilbert space has a countable orthonormal basis, which we claim is a Schauder basis.

**Proof:** Existence:

$$x = \sum_n \langle x, u_n \rangle u_n$$
Uniqueness: If 
\[ x = \sum_n a_n u_n \]
then 
\[ \sum_n (\langle x, u_n \rangle - a_n)u_n = 0. \]

Apply inner product with \( u_m \): by continuity of inner product,
\[ \langle x, u_m \rangle - a_m = \langle \sum_n (\langle x, u_n \rangle - a_n)u_n, u_m \rangle = \langle 0, u_m \rangle = 0 \]
and so each \( a_m = \langle x, u_m \rangle \).

Example: For \( 1 \leq p < \infty \), \( \ell^p \) has a Schauder basis, namely the standard basis, \( \{e_n : n \in \mathbb{N}\} \)

Proof:
Existence: \( x = (x_1, x_2, \ldots) = \sum_{n=1}^{\infty} x_n e_n \) in \( \ell^p \), because 
\[ \|x - \sum_{n=1}^{N} x_n e_n\|_p = \|(0, \ldots, 0, x_{N+1}, x_{N+2}, \ldots)\|_p = (\sum_{n=N+1}^{\infty} |x_n|^p)^{1/p} \rightarrow 0, \]

Uniqueness: Suppose 
\[ x = \sum_{n=1}^{\infty} a_n e_n \]
in \( \ell^p \). Suppose for some \( m \), \( x_m \neq a_m \). then for all \( N \geq m \)
\[ 0 \neq |x_m - a_m| \leq (\sum_{n=1}^{N} |x_n - a_n|^p + \sum_{n=N+1}^{\infty} |x_n|^p)^{1/p} = \|x - \sum_{n=1}^{N} a_n e_n\|_p \]
But the RHS \( \rightarrow 0 \) as \( N \rightarrow \infty \), a contradiction. \( \square \)

\( \ell^p \) is separable because one can show that 
\[ \{q_1, q_2, \ldots, q_n, 0, 0, \ldots : q_n \in \mathbb{Q}\} \]
is a countable dense set.

In fact, separability is a necessary condition for a Banach space to have a Schauder basis.

Theorem: If a Banach space has a Schauder basis, then it is separable.

Proof: Given a Schauder basis \( \{v_1, v_2, \ldots \} \) for a Banach space \( X \) we claim that
\[
\left\{ \sum_{n=1}^{N} q_n v_n : N \in \mathbb{N}, q_n \in \mathbb{Q} \right\}
\]
is dense in \( X \). To see this, let \( x \in X \) and so for some sequence \( a_n \)
\[x = \sum_n a_n v_n.\]
Given \( \epsilon > 0 \), for some \( N \)
\[||x - \sum_{n=1}^{N} a_n v_n|| < \epsilon\]
For each \( n = 1, \ldots, N \), choose \( q_n \in \mathbb{Q} \) s.t.
\[|a_n - q_n| < \epsilon/(N||v_n||)\]
Then
\[|| \sum_{n=1}^{N} a_n v_n - \sum_{n=1}^{N} q_n v_n || \leq \sum_{n=1}^{N} |a_n - q_n| ||v_n|| < \epsilon\]
So, by triangle inequality,
\[||x - \sum_{n=1}^{N} q_n v_n|| < 2\epsilon\]
So, \( X \) is separable. \( \square \)

Example: \( \ell^\infty \) is not separable and thus does not have a Schauder basis.
Proof:

Step 1: It suffices to find an uncountable set \( A \subset \ell^\infty \) s.t. for each \( x, x' \in A \) with \( x \neq x' \), \( ||x - x'|| = 1 \).

Proof: Let \( S \) be a dense set. The balls \( B_{1/2}(x), x \in A \) are disjoint. Then each of these balls would contain at least one element of \( S \) and so these elements would be distinct. But since there are uncountably many balls, there are uncountably many points in \( S \). Thus, \( S \) is uncountable and so \( \ell^\infty \) is not separable.

Step 2: Let \( A = \{(x_1, x_2, \ldots, ) : x_i = 0 \text{ or } 1\} \). This is (pretty much) in 1-1 correspondence with points in the unit interval and so is uncountable. And if \( x, x' \in A \) \( x \neq x' \), then \( ||x - x'|| = 1 \). \( \square \)

Q (Banach, 1930’s): Does every separable Banach space have a Schauder basis?