Lecture 24:

Recall:

Defn: A subset A of a topological space is *nowhere dense* if $\mathrm{int}(\overline{A}) = \emptyset$, equivalently if $\overline{A}$ contains no open set.

Corollary (of Baire Category Theorem): A complete metric space *cannot* be written as a countable union of nowhere dense sets.

Theroem: A Hamel basis for a Banach space must be either finite or uncountable.

Proof: By contradcition. Suppose that a Banach space X had a countably infinite Hamel basis $B := \{v_1, v_2, \ldots\}$.

Write

$$X = \cup X_n$$

where each $X_n := span(v_1, \ldots, v_n\}$.

We will show:

a. each X_n is closed in X

b. each $\mathrm{int}(X_n) = \emptyset$,

contradicting BCT.

a. Follows from finite dimensionality of X_n: complete and therefore closed in X.

b. Let $x \in X_n$ and $\epsilon > 0$. Then $y := x + \epsilon v_{n+1} \notin X_n$, but $y \in B_{2\epsilon||v_{n+1}||}(x)$ and so $B_{2\epsilon||v_{n+1}||}(x) \not\subset X_n$ and so $\mathrm{int}(X_n) = \emptyset$ $\square$

Note that the NVS, c_c which is not complete, has a countable basis (the standard basis vectors, e_i which is 1 in position i and 0 elsewhere).

But the standard basis cannot be a Hamel basis for the Banach space c_0 because it can only generate sequences. with finitely many non-zeros.

A more useful basis:

Defn: A *Schauder basis* for a NVS X (in particular Banach X) is a countable set $\{v_n\}$ s.t. every $x \in X$ can be *expressed uniquely* as

$$x = \sum_n a_n v_n$$

meaning $||x - \sum_{n=1}^N a_n v_n|| \to 0$.

Note that because of uniqueness any Schauder basis is linearly independent.

Clearly, any linear transformation on a vector space is determined by its values on a Hamel basis.

Claim: any continuous linear transformation (BLT) on a NVS is determined by its values on a Schauder basis.

Proof: Let $x = \sum_n a_n v_n \in X$, i.e., $||x - \sum_{n=1}^N a_n v_n|| \to 0$.

Then by continuity of the norm, $||Tx - \sum_{n=1}^N a_n T v_n|| \to 0$ and so $Tx = \sum_n a_n T v_n \in X$. $\square$

So, from the point of view of Banach spaces, Schauder bases are much more useful than Hamel bases.

Which Banach spaces have Schauder bases?

Example: any *separable* Hilbert space has a Schauder basis.

Recall that any separable Hilbert space has a countable orthonormal basis, which we claim is a Schauder basis.

Proof: Existence:

$$x = \sum_n \langle x, u_n \rangle u_n$$

Uniqueness: If

$$x = \sum_n a_n u_n$$

then

$$\sum_n (\langle x, u_n\rangle - a_n) u_n = 0.$$

Apply inner product with u_m: by continuity of inner product,

$$\langle x, u_m\rangle - a_m = \langle \sum_n (\langle x, u_n\rangle - a_n) u_n, u_m\rangle = \langle 0, u_m\rangle = 0$$

and so each $a_m = \langle x, u_m\rangle$.

Example: For $1 \leq p < \infty$, ℓ^p has a Schauder basis, namely the standard basis, $\{e_n : n \in \mathbb{N}\}$

Proof:

Existence: $x = (x_1, x_2, \ldots) = \sum_{n=1}^{\infty} x_n e_n$ in ℓ^p, because

$$||x - \sum_{n=1}^{N} x_n e_n||_p = ||(0, \ldots, 0, x_{N+1}, x_{N+2}, \ldots||_p = (\sum_{n=N+1}^{\infty} |x_n|^p)^{1/p} \to 0,$$

Uniqueness: Suppose

$$x = \sum_{n=1}^{\infty} a_n e_n$$

in ℓ^p. Suppose for some m, $x_m \neq a_m$. then for all $N \geq m$

$$0 \neq |x_m - a_m| \leq (\sum_{n=1}^{N} |x_n - a_n|^p + \sum_{n=N+1}^{\infty} |x_n|^p)^{1/p} = ||x - \sum_{n=1}^{N} a_n e_n||_p$$

But the RHS $\to 0$ as $N \to \infty$, a contradiction. $\Box$

ℓ^p is separable because one can show that

$$\{q_1, q_2, \ldots, q_n, 0, 0, \ldots) : \; q_n \in \mathbb{Q}\}$$

is a countable dense set.

In fact, separability is a necessary condition for a Banach space to have a Schauder basis.

Theorem: If a Banach space has a Schauder basis, then it is separable.

Proof: Given a Schauder basis $\{v_1, v_2, \ldots\}$ for a Banach space X we claim that

$$\{\sum_{n=1}^{N} q_n v_n : N \in \mathbb{N}, q_n \in \mathbb{Q}\}$$

is dense in X. To see this, let $x \in X$ and so for some sequence a_n $x = \sum_n a_n v_n$. Given $\epsilon > 0$, for some N

$$||x - \sum_{n=1}^{N} a_n v_n|| < \epsilon$$

For each $n = 1, \ldots, N$, choose $q_n \in \mathbb{Q}$ s.t.

$$|a_n - q_n| < \epsilon/(N||v_n||)$$

Then

$$||\sum_{n=1}^{N} a_n v_n - \sum_{n=1}^{N} q_n v_n|| \leq \sum_{n=1}^{N} |a_n - q_n| \, ||v_n|| < \epsilon$$

So, by triangle inequality,

$$||x - \sum_{n=1}^{N} q_n v_n|| < 2\epsilon$$

So, X is separable. $\square$

Example: ℓ^∞ is not separable and thus does not have a Schauder basis.

Proof:

Step 1: It suffices to find an uncountable set $A \subset \ell^\infty$ s.t. for each $x, x' \in A$ with $x \neq x'$, $||x - x'|| = 1$.

Proof: Let S be a dense set. The balls $B_{1/2}(x), x \in A$ are disjoint. Then each of these balls would contain at least one element of S and so these elements would be distinct. But since there are uncountably many balls, there are uncountably many points in S. Thus, S is uncountable and so ℓ^∞ is not separable.

Step 2: Let $A = \{(x_1, x_2, \ldots,) : x_i = 0 \text{ or } 1\}$. This is (pretty much) in 1-1 correspondence with points in the unit inteval and so is uncountable. And if $x, x' \in A$ $x \neq x'$, then $||x - x'|| = 1$. $\square$

Q (Banach, 1930's): Does every separable Banach space have a Schauder basis?

A (Enflou, 1970): No.