Lecture 23:

Baire Category Theorem

not intrinsic

Recall:

Defn: A subset U of a topological space X is *dense* if $U \cap W \neq \emptyset$ for all nonempty open sets W. (open in X)

In a metric space, this means that $U \cap B_\epsilon(x) \neq \emptyset$ for each x, ϵ.

Prop: In any topological space the finite intersection of open dense sets is open and dense, and in particular nonempty.

Proof: It is open because the intersection of finitely many open sets is open.

Prove density by induction.

For $n = 1$, this is given. Assume $U_1, \ldots, U_{n+1}$ are open and dense and $\cap_{i=1}^n U_i$ is open and dense. Let W be any nonempty open set. By density $U_{n+1} \cap W$ is nonempty and open, and so

$$(\cap_{i=1}^{n+1} U_i) \cap W = (\cap_{i=1}^n U_i) \cap (U_{n+1} \cap W) \neq \emptyset. \quad \square$$

Baire Category Theorem: Let X be a complete metric space. Then the countable intersection of open dense sets is dense, and in particular non-empty.

Proof:

Let U_n be a sequence of open dense sets. Let W be a nonempty open set. We must show that $\cap_n U_n$ intersects W. We will find a point in the intersection as the limit of an inductively defined Cauchy sequence.

$U_1 \cap W$ is nonempty and open and so there exist x_1 and $\epsilon_1 > 0$, such that

$$\overline{B_{\epsilon_1}(x_1)} \subset U_1 \cap W. \tag{5}$$

Now,,since $U_2 \cap B_{\epsilon_1}(x_1)$ is nonempty and open and so we can choose x_2 and $\epsilon_2 > 0$ arbitrarily small s.t. $\overline{B_{\epsilon_2}(x_2)} \subset U_2 \cap B_{\epsilon_1}(x_1)$.

Thus, inductively we can find x_n and ϵ_n such that

$$\overline{B_{\epsilon_n}(x_n)} \subset U_n \cap B_{\epsilon_{n-1}}(x_{n-1}) \tag{6}$$

Since $\{\overline{B_{\epsilon_n}(x_n)}\}$ is a nested decreasing sequence, for each N and all $n \geq N$, $x_n \in \overline{B_{\epsilon_N}(x_N)}$. Thus, if we choose $\epsilon_n \to 0$, then x_n is Cauchy and thus $x_n \to x$ for some $x \in X$,

We claim that $x \in \cap_n U_n \cap W$.

Since $\{\overline{B_{\epsilon_n}(x_n)}\}$ is a nested sequence of closed sets and $x_n \to x$, we have $x \in \overline{B_{\epsilon_n}(x_n)}$ for all n: given any fixed N, for all $n \geq N$, $x_n \in \overline{B_{\epsilon_N}(x_N)}$ and so $x \in \overline{B_{\epsilon_N}(x_N)}$ for all n. In particular by (6), $x \in U_n$ for all n. And by (5), $x \in W$. So, $\cap_n U_n$ is dense. $\square$

Example: in a complete metric space, a countable intersection of open dense sets need not be open:

The set of irrationals.

In $\mathbb{R}$, the complement of a single point is open and dense. So, the set of irrational numbers is a countable intersection of complements of one point sets, namely the rationals, and so is the countable intersection of open dense sets.

But it is not open. $\square$

Complement of finite set ~~Complement of Cantor set~~

An open dense set is a "very fat, pervasive" set. A countable intersection of open dense sets is a "fairly fat, pervasive" set.

Defn: A subset of a topological space is *residual* if it contains a countable intersection of open dense sets.

Irrationals

Example of residual set: the set of irrationals

Fact: A countable intersection of residual sets is residual.

Application of BCT: Show existence of something w/ cntbly many properties s.t. (of each property is open + dense)

Not intrinsic

Nonempty

Defn: A subset A of a topological space is *nowhere dense* if its closure has empty interior – equivalently it contains no open set.

Examples: finite sets (in $\mathbb{R}$) and Cantor sets (in $\mathbb{R}$) are nowhere dense.

Neither the set of rationals nor the set of irrationals is nowhere dense.

Prop: A set is open and dense iff its complement is closed and nowhere dense.

(think of a finite set or a Cantor set).

Proof:

U is open and dense $\iff$

U is open and for every nonempty open W, $U \cap W \neq \emptyset \iff$

U^c is closed and no open W is contained in $U^c \iff$

U^c is closed and nowhere dense. $\square$

Corollary (of Baire Category Theorem): A complete metric space cannot be written as a countable union of nowhere dense sets.

Proof: Suppose not. Then $X = \cup_n E_n$ where the closure of E_n has empty interior. $\Rightarrow X = \cup_n \overline{E_n}$

Then $(\overline{E_n})^c$ is open and dense. But then

$$\emptyset = X^c = \cap_n (\overline{E_n})^c$$

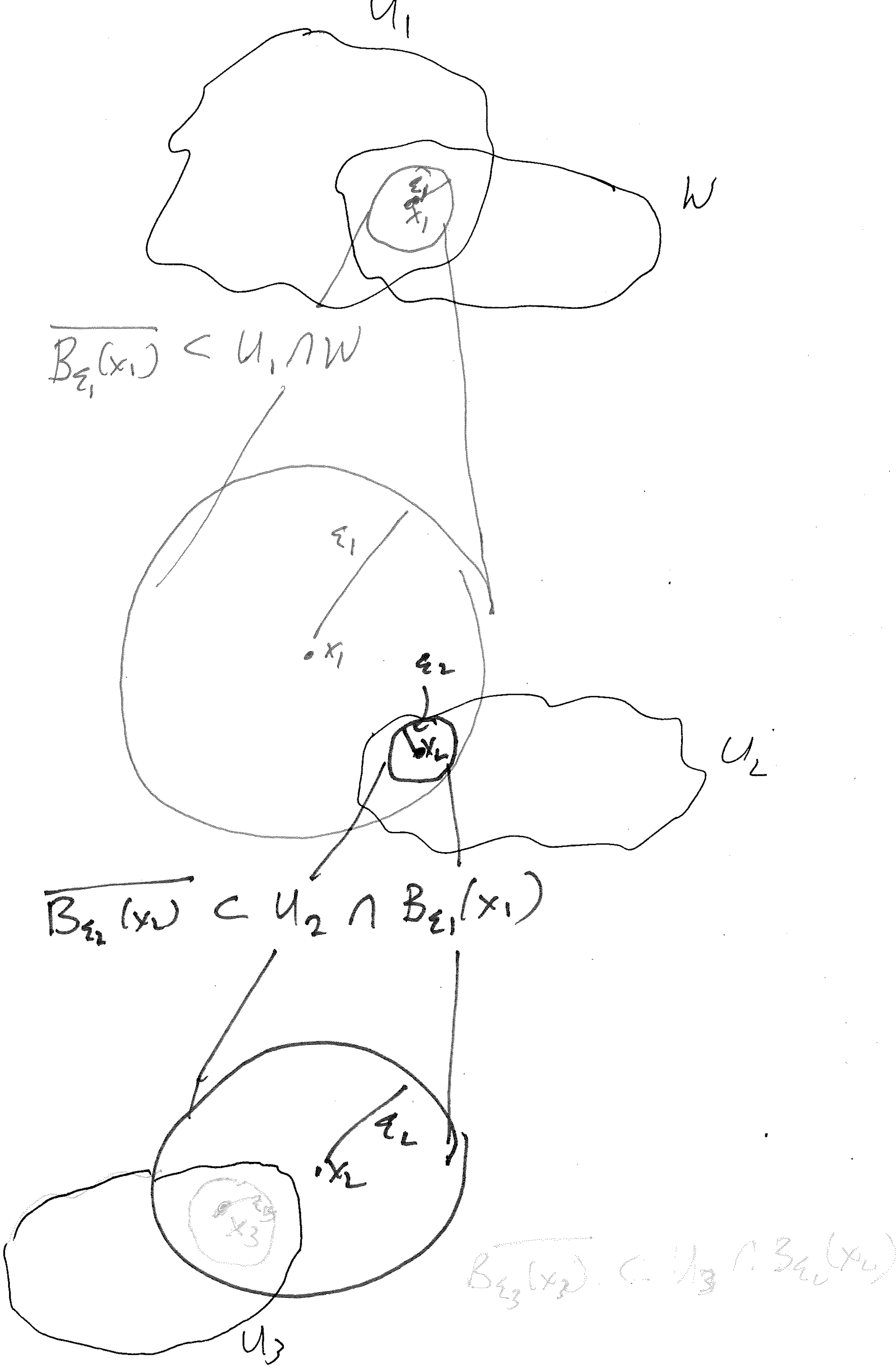
contradicting BCT. $\square$

Defn: A set is *meager* if it is a countable union of nowhere dense sets.

Example of a meager set: the set of rationals is meager but not nowhere dense.

Exercise: A set is meager iff its complement is residual.

Theroem: A Hamel basis for a Banach space must be either finite or uncountable.



U_1
W
ε_1
x_1
$\overline{B_{\varepsilon_1}(x_1)} \subset U_1 \cap W$
ε_1
x_1
ε_2
x_2
U_2
$\overline{B_{\varepsilon_2}(x_2)} \subset U_2 \cap B_{\varepsilon_1}(x_1)$
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$\overline{B_{\varepsilon_3}(x_3)} \subset U_3 \cap B_{\varepsilon_2}(x_2)$
U_3