Lecture 21:

Review:

Normed Vector Space (NVS)

Banach space: NVS + complete

Inner product space (IPS)

Hilbert space: IPS + complete

IPS $\subset$ NVS

Hilbert $\subset$ Banach

Homeomorphic isomorphism: Vector space isomorphism + homeomorphism (HW2#4)

Isometric isomorphism: Vector space isomorphism + norm-preserving (HW2#6)

Isometric isomorphism $\Rightarrow$ Homeomorphic isomorphism

Example: c_0 and c are Homeomorphically isomorphic but not Isometrically isomorphic, (HW2#4, HW3#8).

Properties of Banach spaces:

– absolute convergence criterion for completeness (Folland, Theorem 5.1)

– a subspace of Banach space is Banach iff subspace is closed

– non-compactness of unit ball (unless finite dimensional)

Properties of Hilbert spaces:

– Distance from point to closed convex subset is achieved uniquely false for Banach, HW4 #5)

– For a closed subspace W of H, Existence of orthogonal complement $W^\perp$ and orthogonal decomposition $H = W \oplus W^\perp$

– Orthonormal bases always exist;: 4 equivalent criteria, for an o.n. set to be an o.n. basis inclduing:

$\overline{\text{span}(\{u_\alpha\})} = H$

$x = \sum_\alpha \langle x, u_\alpha \rangle u_\alpha$

– Orthonormal basis is countable iff space is separable

Hamel basis (ordinary basis)

—- exists for any nonzero vector space

—- for a Banach space either finite or uncountable (will prove soon).

—- is there a better basis for Banach spaces?

An NVS is an IPS iff Parallelogragm Law holds.

Completion of NVS to a Banach space: HW1,2,3, problem 7,7,7.

Examples of Banach: $\mathbb{R}^n, \mathbb{C}^n$, $B(\Omega)$, $BC(\Omega)$, $C(\Omega)$, $C_0(\Omega)$, $C_c(\Omega)$, $C^1(\Omega)$ ℓ^∞, ℓ^p, c, c_0, L^p, L^∞.

Examples of NVS that are not Banach: c_c, C^1 as subspace of $C([0,1])$, $C([0,1])$ as subspace of $L^1([0,1])$

Examples of Hilbert: L^2, ℓ^2

Standard models:

—- any separable Banach space sits in (isometric isomorphism embedding) ℓ^∞ (HW3 #2)

—- any separable Hilbert space is isomeric isomorphic to ℓ^2.

—- Note $\ell^2 \subset \ell^\infty$

BLTs: Linear transformation from one NVS X to another NVS Y that is

– continuous iff bounded

$L(X,Y)$: set of BLTs from X to Y

– $L(X,Y)$ is an NVS with operator norm
– If Y is Banach (e.g., $\mathbb{R}, \mathbb{C}$), then $L(X,Y)$ is Banach

BLFs: BLTs from X to K

$X^* := L(X,K)$, the dual space

X^* is always Banach

Characterization of X^* for various X:

– $(L^p)^* = L^q$ $(1 \leq p < \infty)$ (Riesz rep)
– $(L^2)^* = L^2$ (Riesz-Fischer rep)

– $C([0,1])^*$ is space of finite signed masures on $[0.1]$.

– $c_0^* = \ell^1$

If H is Hilbert so is H^*; H is conjugate-linear isometric isomorphic to H^*.

Hahn-Banach: existence of lots of linear functionals

—– a purely linear algebra statement (i.e., no topology) but assumes a sublinear functional.

—– Minkowski functional (HW3, #4) used to prove hyperplane separation theorems

Double dual spaces X^{**}: dual of the dual

Canonical embedding: $X \to X^{**}$; $x \mapsto \hat{x}$ where $\hat{x}$ is the evaluation functional.

— is always an injective isometric isomorphism.

Defn: *Reflexive* Banach spaces: $x \mapsto \hat{x}$ is surjective, and in partiuclar $X^{**} = X$

– Examples of reflexive Banach spaces: L^p $(1 < p < \infty)$, Hilbert spaces

– BLFs on reflexive Banach spaces are norm-attaining (HW2#9b)