Lecture 21:
Review:
Normed Vector Space (NVS)
Banach space: NVS + complete
Inner product space (IPS)
Hilbert space: IPS + complete
IPS ⊂ NVS
Hilbert ⊂ Banach
Homeomorphic isomorphism: Vector space isomorphism + homeomorphism (HW2#4)
Isometric isomorphism: Vector space isomorphism + norm-preserving (HW2#6)
Isometric isomorphism ⇒ Homeomorphic isomorphism
Example: $c_0$ and $c$ are Homeomorphically isomorphic but not Isometrically isomorphic, (HW2#4, HW3#8).
Properties of Banach spaces:
– absolute convergence criterion for completeness (Folland, Theorem 5.1)
– a subspace of Banach space is Banach iff subspace is closed
– non-compactness of unit ball (unless finite dimensional)
Properties of Hilbert spaces:
– Distance from point to closed convex subset is achieved uniquely false for Banach, HW4 #5)
– For a closed subspace $W$ of $H$, Existence of orthogonal complement $W^\perp$ and orthogonal decomposition $H = W \oplus W^\perp$
– Orthonormal bases always exist; 4 equivalent criteria, for an o.n. set to be an o.n. basis including:

\[ \text{span}(\{u_\alpha\}) = H \]

\[ x = \sum_\alpha \langle x, u_\alpha \rangle u_\alpha \]

– Orthonormal basis is countable iff space is separable

Hamel basis (ordinary basis)

— exists for any nonzero vector space

— for a Banach space either finite or uncountable (will prove soon).

— is there a better basis for Banach spaces?

An NVS is an IPS iff Parallelogram Law holds.

Completion of NVS to a Banach space: HW1,2,3, problem 7,7,7.

Examples of Banach: \( \mathbb{R}^n, \mathbb{C}^n, B(\Omega), BC(\Omega), C(\Omega), C_0(\Omega), C_c(\Omega), C^1(\Omega) \) \( \ell_\infty, \ell^p, c, c_0, L^p, L^\infty \).

Examples of NVS that are not Banach: \( c_c, C^1 \) as subspace of \( C([0, 1]), C([0, 1]) \) as subspace of \( L^1([0, 1]) \)

Examples of Hilbert: \( L^2, \ell^2 \)

Standard models:

— any separable Banach space sits in (isometric isomorphism embedding) \( \ell^\infty \) (HW3 #2)

— any separable Hilbert space is isometric isomorphic to \( \ell^2 \).

— Note \( \ell^2 \subset \ell^\infty \)

BLT: Linear transformation from one NVS \( X \) to another NVS \( Y \) that is

– continuous iff bounded
$L(X, Y)$: set of BLTs from $X$ to $Y$

- $L(X, Y)$ is an NVS with operator norm
- If $Y$ is Banach (e.g., $\mathbb{R}, \mathbb{C}$), then $L(X, Y)$ is Banach

BLFs: BLTs from $X$ to $K$

$X^* := L(X, K)$, the dual space

$X^*$ is always Banach

Characterization of $X^*$ for various $X$:

- $(L^p)^* = L^q$ (1 ≤ $p < \infty$) (Riesz rep)
- $(L^2)^* = L^2$ (Riesz-Fischer rep)
- $C([0, 1])^*$ is space of finite signed measures on [0,1].
- $c_0^* = \ell^1$

If $H$ is Hilbert so is $H^*$; $H$ is conjugate-linear isometric isomorphic to $H^*$.

Hahn-Banach: existence of lots of linear functionals

— a purely linear algebra statement (i.e., no topology) but assumes a sublinear functional.

— Minkowski functional (HW3, #4) used to prove hyperplane separation theorems

Double dual spaces $X^{**}$: dual of the dual

Canonical embedding: $X \to X^{**}; x \mapsto \hat{x}$ where $\hat{x}$ is the evaluation functional.

— is always an injective isometric isomorphism.

Defn: Reflexive Banach spaces: $x \mapsto \hat{x}$ is surjective, and in particular $X^{**} = X$

- Examples of reflexive Banach spaces: $L^p$ (1 < $p < \infty$), Hilbert spaces
– BLFs on reflexive Banach spaces are norm-attaining (HW2#9b)