Lecture 20:

Recall:

Riesz-Frechet representation theorem:

Let H be a Hilbert space. Then $f \in H^*$ iff there exists $y \in H$ s.t. for all $x \in H$

$$f(x) = \langle x, y \rangle$$

Moreover, given f, y is unique.

Defn: Define $y^* \in H^*$ by $y^*(x) = \langle x, y \rangle$.

So, $H^* = \{y^* : y \in H\}$ by Riesz-~~Fischer~~. Frechet

HW4: Prop: $\langle y^*, z^* \rangle_{H^*} := \langle z, y \rangle_H$ is an inner product on H^* whose induced norm is the operator norm on H^*.

Why the reversal?

$$iy^*(x) = i\langle x, y \rangle = \langle x, -iy \rangle$$

Thus, $iy^* = (-iy)^*$ and so

$$\langle iy^*, z^* \rangle = \langle (-iy)^*, z^* \rangle = \langle z, -iy \rangle = i\langle z, y \rangle = i\langle y^*, z^* \rangle$$

as it should be.

Regarding the norm,

$$\langle y^*, y^* \rangle_{H^*} = \langle y, y \rangle_H = ||y||_H^2 = ||y^*||_{H^*}^2$$

where the latter quantity is the operator norm and the latter equality follows from the "If" part of the Riesz-Frechet Theorem. $\square$

Defn: A *conjugate-linear functional* is a map $T : X \to K$ on a vector space X that satisfies $T(ay + bz) = \overline{a}T(y) + \overline{b}T(z)$.

Prop: The map $D : H \mapsto H^*$ defined by $D(y) = y^*$ is a conjugate-linear isometric isomorphism of H onto H^*. Moreover, $\langle D(y), D(z) \rangle_{H^*} = \overline{\langle y, z \rangle_H}$.

Proof:

Conjugate-Linearity: follows from sesqui-linearity of the inner product:

$$(D(ay+bz))(x) = \langle x, ay+bz\rangle = \overline{a}\langle x, y\rangle + \overline{b}\langle x, z\rangle = \overline{a}D(y)(x) + \overline{b}D(y)($$

Norm-preserving: $||y^*||_{H^*} = ||y||_H$ by the proof of the "If" part of Riesz-Fischer.

Injective: Follows from norm-preserving, as usual.

Surjective: Follows from "Only If" part of Riesz-Fischer.

The Moreover is an immediate consequence of the definition of D. $\square$

Prop:

1. $D \circ D : H \to H^{**}$ is the canonical embedding
2. $D \circ D$ is a (surjective) isometric isomorphism.
3. Any Hilbert space is reflexive.

Proof: 1. For $y \in H, z^* \in H^*$,

$$(D \circ D)(y)(z^*) = (D(y^*))(z^*) = \langle z^*, y^*\rangle_{H^*} = \langle y, z\rangle_H$$

Letting $J(y) = \hat{y}$ denote the canonical embedding, we have

$$J(y)(z^*) = \hat{y}(z^*) = z^*(y) = \langle y, z\rangle_H$$

So, $\hat{y} = J(y) = D \circ D(y)$.

2. $D \circ D$ is the composition of two conjugate-linear isometric isomorphisms and thus is a an isometric isomorphism, in particular surjective.

3. The canonical embedding is surjective. $\square$

Theorem: "All Hilbert spaces look like ℓ^2" More precisely:

Let H be a Hilbert space and $\{u_\alpha\}_{\alpha \in A}$ be an orthonormal basis for H. Then H is isometrically isomorphic to $\ell^2(A)$.

Proof: Let $\Phi : H \to \ell^2(A)$ be defined by

$$\Phi(x)(\alpha) = \langle x, u_\alpha \rangle$$

Check $\Phi(x) \in \ell^2(A)$:
By Parseval,

$$||\Phi(x)||_2^2 = \sum_\alpha |\Phi(x)(\alpha)|^2 = \sum_\alpha |\langle x, u_\alpha \rangle|^2 = ||x||^2 < \infty$$

Check Linearity:

$$\Phi(ax+by)(\alpha) = \langle ax+by, u_\alpha \rangle = a\langle x, u_\alpha \rangle + b\langle y, u_\alpha \rangle = a\Phi(x)(\alpha) + b\Phi(y)(\alpha)$$

Check Norm-preserving: By Parseval's identity

Check Injective: follows from norm-preserving

Check Surjective:

Given $f \in \ell^2(A)$, first recall that the support of f, i.e. $\{\alpha : f(\alpha) \neq 0\}$, is countable, and so we can order the support of f as $\alpha_1, \alpha_2, \ldots$ (the support may change with f).

Since $\sum_i |f(\alpha_i)|^2 < \infty$, the partial sums of the series are Cauchy in $\mathbb{R}$. So, by the Pythagorean theorem,

$$||\sum_{i=m}^{n} f(\alpha_i) u_{\alpha_i}||^2 = \sum_{i=m}^{n} |f(\alpha_i)|^2$$

the partial sums of the series $\sum_i f(\alpha_i) u_{\alpha_i}$ are Cauchy in H and thus that series converges to some

$$x = \sum_i f(\alpha_i) u_{\alpha_i}$$

By continuity of the inner product and orthonormality of the basis, we see that for each j

$$\langle x, u_{\alpha_j} \rangle = \langle \sum_i f(\alpha_i) u_{\alpha_i}, u_{\alpha_j} \rangle = f(\alpha_j)$$

Thus, $\Phi(x) = f$. $\square$