HW3: Due Friday

Lecture 19:

Notes from last time: argument for showing that complex exponentials form an o.n. basis:

– distinction between sup norm and ℓ^2 norm

– switching from $[0, 1]$ to unit circle in the middle of proof.

Recall:

Defn: A metric space is *separable* if it has a countable dense set.

Examples:

1. $L^2([0, 1], \mu)$, with μ Lebesgue, is separable: simple functions with complex rational coefficients and intervals with rational endpoints, are dense.

2. $\ell^2(X)$ is separable iff X is countable.

Proof: Assume countable, then the standard basis vectors form a countable dense set.

Assume separable, so there is a countable dense set $D = \{f_n : n \in \mathbb{N}\}$. For each n, let $U_n = \{\alpha \in X : f_n(\alpha) \neq 0\}$. Since each f_n is square summable, each U_n is countable and so $U = \cup_n U_n$ is countable.

If X were uncountable, then there exists $\alpha \in X \setminus U$. Then for any n ~~$g = 1_{\{\alpha\}}$ and any n $\|g - f_n\| \geq 1$~~ $\|1_{\{\alpha\}} - f_n\| \geq 1$. Contradiction to density of D.

□

Theorem: Let H be a Hilbert space. TFAE

1. H is separable
2. H has a countable orthonormal basis
3. Every orthonormal basis of H is countable

Proof:

1 implies 2: Let $\{x_n\}$ be a countable dense subset of H.

We construct from $\{x_n\}$ a countable orthonormal basis in two steps:

Step 1:
Iteratively delete any x_n that is in the span of the previous elements. Call the resulting sequence $\{y_n\}$. If any finite subset of $\{y_n\}$ were linearly dependent, then choose the smallest N s.t. $y_1, \dots, y_N$ is linearly dependent; but then y_N should have been deleted.

The set $\{y_n\}$ is linearly independent, and it has same linear span as the countable dense set and so its linear span is dense.

Step 2: construct, by Gram-Schmidt orthonormalization, from $\{y_n\}$ an orthonormal sequence $\{u_n\}$ whose linear span is the same as $\{y_n\}$ and thus dense in H and so is an orthonormal basis (by characterization 1 of orthonormal basis).

Define $u_1 = \frac{y_1}{||y_1||}$ and inductively for each $N > 1$,

$$z_N = y_N - \sum_{n=1}^{N-1} \langle y_N, u_n \rangle u_n, \quad u_N = \frac{z_N}{||z_N||}$$

2 implies 1: the set of all complex rational linear combinations of elements of an orthonormal basis $\{u_n\}$ is a countable set whose closure is the same as the linear span of $\{u_n\}$ and is therefore dense in H.

3 implies 2: obvious

2 implies 3: Let $\{x_n\}$ be a countable orthonormal basis. Let $\{y_a\}_{a \in A}$ be another orthonormal basis. Let

$$R_n \ \cancel{A_n} := \{a \in A : \langle x_n, y_a \rangle \neq 0\}.$$

Applying Bessel's inequality to write x_n in terms of the basis $\{y_a\}_{a \in A}$, we obtain that each $\cancel{A_n}$ is countable.

R_n

x_1 y_a y_b y_c ...

x_2

Injection from Column Indices $\hookrightarrow$ Countable set

If for some $a \in A$, each $< x_n, y_a >= 0$, then by completeness of the basis $\{x_n\}$, $y_a = 0$ and thus cannot be part of an orthonormal set. It follows that each $a \in A$ must belong to some A_n. So, $A = \cup A_n$ and thus must be countable. $\square$

Riesz-Freschet representation theorem:

Let H be a Hilbert space. Then $f \in H^*$ iff there exists $y \in H$ s.t. for all $x \in H$

$$f(x) = \langle x, y \rangle$$

Recall $H^* = \{BLT: H \to K\}$

Moreover, given f, y is unique.

Proof: "If:" $f(x)$ as defined is clearly a linear functional and it is bounded: by the C-S inequality:

$$|f(x)| = |\langle x, y \rangle| \leq ||y|| \; ||x||$$

and so $||f|| \leq ||y||$ (in fact, since $f(y) = ||y||^2$, we have $||f|| = ||y||$).

Note: It is important that we write $f(x)$ as above, rather than $\langle y, x \rangle$, so that $f(x)$ is a linear functional.

"Only if:" If $f = 0$, then $y = 0$ will do.

If not, $W := Ker(f)$ is a proper closed subpsace of H (it is closed because f is continuous). Thus, there exists $z \in W^\perp$ s.t. $||z|| = 1$.

For given $x \in H$, let $u = f(x)z - f(z)x$. Then $u \in W$. So,

$$0 = \langle u, z \rangle = f(x)||z||^2 - f(z)\langle x, z \rangle = f(x) - \langle x, \overline{f(z)}z \rangle$$

Thus, $y = \overline{f(z)}z$ will do.

Uniqueness: if for all x, $\langle x, y \rangle = \langle x, y' \rangle$ and so $\langle x, y - y' \rangle = 0$, then setting $x = y - y'$, we get $y - y' = 0$. $\square$

Recall that $(L^p)^* = L^q$ where p and q are conjugate exponents, meaning that there is an isometric isomorphism from L^q onto $(L^p)^*$.