Lecture 18:

Recall:

Let H be a Hilbert space and E an orthonormal *set* in H, i.e., orthogonal and length $= 1$.

TFAE:

$\mathbb{R}^2$ $\{(1,0),(1,1)\}$

1. $\overline{\text{span}(E)} = H$ ("closed linear span"), i.e., the linear span of E is dense in H
2. (Completeness of basis) If $< x, u_\alpha >= 0$ for all α, then $x = 0$.

$\mathbb{R}^1$: $\{\frac{1}{2}\}$

3. For each $x \in H$, $x = \sum_{\alpha \in A} < x, u_\alpha > u_\alpha$ and this series is unconditionally convergent.
4. (Parseval's Identity; Bessel's inequality with equality) for all $x \in H$, $||x||^2 = \sum_{\alpha \in A} | < x, u_\alpha > |^2$.

If E satisfies any one of these equivalent conditions it is called an *orthonormal basis.*

Theorem: Every nonzero Hilbert space has an orthonormal basis.

Proof: Order the orthonormal subsets of H by inclusion. Every totally ordered subset has an upper bound, namely its union. By Zorn, it has a maximal element E. Let $W := \overline{\text{span}(E)}$ which is closed linear subspace.

$H = W \oplus W^\perp$

If E is not an orthonormal basis, then $W \neq H$. Then $W^\perp \neq \{0\}$. For any nonzero $y \in W^\perp$, $E \cup \{y/||y||\}$ is an orthonormal set which properly contains E. This contradicts the maximality of E, and so $W = H$ and E is an orthonormal basis. $\square$

Examples of orthonormal bases:

1. $L^2([0,1])$, with Lebesgue measure :

$$E := \{e^{2\pi i n\theta} : n \in \mathbb{Z}\}.$$

– Orthogonality: for $n \neq m$,

$$\langle e^{2\pi i n\theta}, e^{2\pi i m\theta}\rangle = \int e^{2\pi i n\theta} e^{-2\pi i m\theta}\, d\theta$$

$$= \int_0^1 (\cos 2\pi(n-m)\theta)) + i(\sin 2\pi(n-m)\theta))d\theta$$

$$= \cancel{i}\frac{1}{2\pi(n-m)} \sin(2\pi(n-m)\theta)|_0^1 \cancel{+} \frac{-i}{2\pi(n-m)} \cos(2\pi(n-m)\theta)|_0^1 = 0.$$

– Length = 1:

$$\langle e^{2\pi i n\theta}, e^{2\pi i n\theta}\rangle = \int_0^1 e^{2\pi i n\theta} e^{-2\pi i n\theta}\, d\theta = \int_0^1 d\theta = 1.$$

– Orthonormal Basis: For $f \in L^2$,

$$\hat{f}(n) := \langle f, e^{2\pi i n\theta}\rangle = \int_0^1 f(\theta) e^{-2\pi i n\theta} d\theta$$

Want characterization 3 of orthonormal basis:

$$f = \sum_n \hat{f}(n) e^{2\pi i n\theta}$$

with convergence in L^2, i.e. $||f - \sum_{-N}^{N} \hat{f}(n)e^{2\pi i n\theta}||_2 \to 0$.

Fourier series representation of L^2 functions.

This is true:

Step 1: $C([0,1])$ is dense in $L^2([0,1])$ in the L^2 metric, i.e. for all $f \in L^2([0,1])$ and $\epsilon > 0$, there is a continuous function g s.t. $||f - g||_2 < \epsilon$.

Step 2: Apply complex Stone-Weirstrass Theorem (Folland, section 4.7): Let X be a compact metric space. Let $\mathcal{A}$ be a subset of $C(X)$ which satisfies:

a. $\mathcal{A}$ is an *algebra*, i.e., a subspace of $C(X)$ that is closed under multiplication of functions

b. $\mathcal{A}$ is closed under complex conjugation

c. $\mathcal{A}$ contains the constant functions

d. $\mathcal{A}$ separates points, i.e., for all $x, y \in X$, $x \neq y$, there exists $f \in A$ s.t. $f(x) \neq f(y)$.

Then $\mathcal{A}$ is dense in $C(X)$ w.r.t. the sup norm.

Apply to:

$$\mathcal{A} := \text{ complex linear span } (E)$$

, i.e., all complex linear combinations of elements of E.

Verify that $\mathcal{A}$ satisfies a, b, c, d:

a,b,c are straightforward; for condition d, $e^{2\pi i x}$ "almost" separates points in $[0, 1]$;

but you can never separate the points $x = 0$ and $x = 1$; so, view the functions in E as functions on the unit circle instead of $[0, 1]$.

Thus, $L^2([0, 1]) = \overline{span(E)}$ (closure in L^2 metric): because $f \in L^2([0, 1])$ is approximated by $g \in C(X)$ which is approximated by a linear combination of E, which is span(E). by Stone-Weierstrass

This is characterization 1 of orthonormal basis, and so E is an orthonormal basis.

2. $\ell^2(X)$, with counting measure. The standard orthonormal basis: $E := \{u_\alpha\}_{\alpha \in X}$.

$$u_\alpha(\beta) := \left\{ \begin{array}{ll} 1 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{array} \right\}$$

$$\langle u_\alpha, u_\gamma \rangle = \int u_\alpha \overline{u_\gamma} d\mu = \sum_\beta u_\alpha(\beta) u_\gamma(\beta)$$

$$= \left\{ \begin{array}{ll} 1 & \alpha = \gamma \\ 0 & \alpha \neq \gamma \end{array} \right\}$$

So, E is an orthonormal set. And it is an orthonormal basis since $\overline{span(E)} = \ell^2(X)$ in ℓ^2 because if $f \in \ell^2(X)$, then it can have only countably many nonzero terms:

$$\sum_{i=1}^{\infty} |f(\alpha_i)|^2 < \infty$$

and approximate this sum by finite linear combinations of E.

If $X = \mathbb{N}$, then E is can be viewed as the standard basis vectors: $(1, 0, 0, 0, \ldots,), (0, 1, 0, 0, \ldots,), \ldots$

Defn: A metric space is *separable* if it has a countable dense set.

Examples:

1. $L^2([0,1], \mu)$, with μ Lebesgue, is separable: simple functions with complex rational coefficients and intervals with rational endpoints, are dense.

2. $\ell^2(X)$ is separable iff X is countable.

Proof: Assume countable, then the standard basis vectors form a countable dense set.

Assume separable, so there is a countable dense set. The support of any element of $\ell^2(X)$ is countable since it is square summable. So, the union of supports of the countable dense set in $\ell^2(X)$ is countable. If X were uncountable, then there is an element of X whose characteristic function cannot be approximated by the countable dense set. Contradiction.