Lecture 17:

Recall:

Theorem: Let M be a closed subpsace of a Hilbert space H. Then $H = M \oplus M^{\perp}$, i.e., each $x \in H$ is uniquely expressible as $y + z$ where $y \in M, z \in M^{\perp}$. In fact, y, z are the unique points in $M, M^{\perp}$ closest to x.

Prop: Let H be a Hilbert space.

1. For any set $E \subseteq H$, $(E^{\perp})^{\perp}$ is the smallest closed subspace that contains E.

2. If M is a closed subspace, then $(M^{\perp})^{\perp} = M$.

Also, if M is a subspace then $(M^{\perp})^{\perp}$ is the closure of M.

Proof: 2 follows from 1 since M is the smallest closed subspace that contains M.

Proof of 1: Will use $A \subseteq B$ implies $A^{\perp} \supseteq B^{\perp}$.

$(E^{\perp})^{\perp}$ is a closed subspace that contains E.

Since an arbitrary intersection of closed sets is closed and an arbitrary intersection of subspaces is a subspace, the smallest closed subspace that contains a given set is the intersection of all closed subspaces that contains the given set.

Let N be the smallest closed subspace that contains E. Then $N \cap (E^{\perp})^{\perp} = N$ and so

$$E \subset N \subset (E^{\perp})^{\perp}$$

Suppose $(E^{\perp})^{\perp} \neq N$. Then there exists $x \in (E^{\perp})^{\perp} \setminus N$.

Since $N \oplus N^{\perp} = H$, we can write $x = y + z, y \in N, z \in N^{\perp}$.

Since $z \in N^{\perp}$, $\langle y, z \rangle = 0$.

Since $E \subset N$, $E^{\perp} \supset N^{\perp}$, $(E^{\perp})^{\perp} \subset (N^{\perp})^{\perp}$, and so x is orthogonal to every element of $N^{\perp}$, in particular, $\langle x, z \rangle = 0$.

Thus, $\langle z, z\rangle = \langle x - y, z\rangle = 0$. Thus, $z = 0$ and so $x = y \in N$, a contradiction. $\square$

Defn: A set $\{u_\alpha\}_{\alpha\in A}$ is *orthonormal* if:

1. each $||u_\alpha|| = 1$.
2. for each $\alpha \neq \beta$, $u_\alpha \perp u_\beta$.

Bessel's inequality: For an orthonormal set $\{u_\alpha\}_{\alpha\in A}$ and any $x \in H$,

$$\sum_{\alpha\in A} | < x, u_\alpha > |^2 \leq ||x||^2$$

Note; A may be uncountable, but by Bessel, for each x, there are only countably many α s.t. $< x, u_\alpha > \neq 0$. (depending on x)

Proof: The sum is, by defintion, the sup of all finite partial sums. So, it suffices to show that for any finite $F \subset A$,

$$\sum_{\alpha\in F} | < x, u_\alpha > |^2 \leq ||x||^2$$

$$||x - \sum_{\alpha\in F} < x, u_\alpha > u_\alpha||^2 = ||x||^2 - 2\Re < x, \sum_{\alpha\in F} < x, u_\alpha > u_\alpha >$$

$$+ ||\sum_{\alpha\in F} < x, u_\alpha > u_\alpha||^2$$

$$= ||x||^2 - 2\sum_{\alpha\in F} | < x, u_\alpha > |^2 + \sum_{\alpha\in F} | < x, u_\alpha > |^2$$

(by the Pythagorean Theorem)

$$= ||x||^2 - \sum_{\alpha\in F} | < x, u_\alpha > |^2$$

Thus,

$$||x||^2 - \sum_{\alpha\in F} | < x, u_\alpha > |^2 \geq 0.$$

$\square$

TFAE: Let E be an orthonormal set in H.

1. $\overline{\text{span}(E)} = H$ ("closed linear span")

2. (Completeness of basis) If $< x, u_\alpha >= 0$ for all α, then $x = 0$.

3. For each $x \in H$, $x = \sum_{\alpha \in A} < x, u_\alpha > u_\alpha$ and this series is unconditionally convergent. and ctbly. many terms

4. (Parseval's Identity; Bessel's inequality with equality) for all $x \in H$, $||x||^2 = \sum_{\alpha \in A} | < x, u_\alpha > |^2$.

If E satisfies any one of these equivalent conditions it is called an *orthonormal basis.*

Proof:

2 implies 3: Enumerate $F_x := \{\alpha :< x, u_\alpha >\neq 0\}$ in any order as $\alpha_1, \alpha_2, \ldots$. By Bessel's inequality $\sum_i | < x, u_{\alpha_i} > |^2$ converges and thus its partial sums are Cauchy. For any $m \leq n$, by Pythagorean theorem,

$$||\sum_{i=m}^{n} < x, u_{\alpha_i} > u_{\alpha_i}||^2 = \sum_{m}^{n} | < x, u_{\alpha_i} > |^2$$

which approaches 0 as m, n get large. Thus, the partial sums of $\sum_i < x, u_{\alpha_i} > u_{\alpha_i}$ are Cauchy and so the series converges to some y since H is complete. Then for any $\beta \in A$,

$$< y-x, u_\beta) =< \sum_i < x, u_{\alpha_i} > u_{\alpha_i} - x, u_\beta >=< x, u_\beta > - < x, u_\beta >= 0.$$

(for middle equality, use continuity of $\langle \cdot, \cdot \rangle$ and argue differently for $\beta \in \{\alpha_i\}$ and $\beta \notin \{\alpha_i\}$).

By 2, $y = x$.

3 implies 4: As in proof of Bessel, for any n,

$$||x||^2 - \sum_1^n |< x, u_{\alpha_i} >|^2 = ||x - \sum_1^n < x, u_{\alpha_i} > u_{\alpha_i}|| \to 0.$$

Thus, 4 holds.

Clearly, 4 implies 2.

So, 2,3 and 4 are all equivalent.

1 implies 2: If $< x, u_\alpha >= 0$ for all α, then $< x, y >= 0$ for all y in the linear span of the u_α. By continuity of the inner product, $< x, y >= 0$ for all y in the closed linear span of the u_α, which, by 1, is all of H. In particular, $< x, x >= 0$ and so $x = 0$.

Clearly, 3 implies 1. □

Q: Is there a way of showing that 3 implies 4 by using fact that any re-ordering of an absolutly convergent series converges always to the same limit?

Theorem: Every Hilbert space has an orthonormal basis.

Proof: Order the orthonormal subsets of H by inclusion. Every totally ordered subset has an upper bound, namely its union. By Zorn, it has a maximal element E. Let $W := \overline{\text{span}(E)}$ which is closed linear subspace.

If E is not an orthonormal basis, then $W \neq H$. Then $W^\perp \neq \{0\}$. For any nonzero $y \in W^\perp$, $E \cup \{y/||y||\}$ is an orthonormal set which properly contains E. This contradicts the maximality of E, and so $W = H$ and E is an orthonormal basis. □