Lecture 16:

Recall: A Banach space X is a complete NVS.

Fact: $X \to [0, \infty)$, $x \mapsto ||x||$ is continuous.

Proof (in fact, works for NVS):

Let $x_n \to x$.

$$| \, ||x_n|| - ||x|| \, | \leq ||x_n - x||$$

Since $||x_n - x|| \to 0$, $||x_n|| \to ||x||$. $\square$

Recall: A Hilbert space is a complete inner product space.

Fact: $X \times X \to K$, $(x, y) \mapsto \langle x, y \rangle$ is continuous.

Proof (in fact, works for inner product spaces): If $x_n \to x, y_n \to y$, then

$$|\langle x, y \rangle - \langle x_n, y_n \rangle| \leq |\langle x, y \rangle - \langle x, y_n \rangle| + |\langle x, y_n \rangle - \langle x_n, y_n \rangle|$$

$$= |\langle x, y - y_n \rangle| + |\langle x - x_n, y_n \rangle|$$

by C-S:

$$\leq ||x|| \, ||y - y_n|| + ||y_n|| \, ||x - x_n||$$

Since Hilbert spaces are Banach spaces, $||x||$ is continuous, and so $||y_n|| \to ||y||$ and so upper bound tends to 0 as $n, m \to \infty$. $\square$

Defn: For a vector space X, a *convex combination* of $x, y \in X$ is a point of the form $tx + (1-t)y$ such that $t \in [0, 1]$. A subset S of X is *convex* if whenever $x, y \in S$ then every convex combination of x, y is in S.

A convex combination is a weighted average.

Intuitively, a convex set is one in which for every pair of points in the set, the entire line segment between the points is contained in the set.

Example: Any subspace of a vector space is convex.

Theorem (closest point): Let K be a nonempty closed convex subset of a Hilbert space H. Let $x \in H \setminus K$. Then $\inf_{y\in K} ||x-y|| > 0$ is achieved uniquely. Since K is closed.

In other words, there is a unique point in K closest to x.

HW4: This is false (both existence and uniqueness) in Banach spaces, even when K is a closed subspace.

Proof: Let $\{y_n\}$ be a sequence in K such that $||x - y_n||$ converges to $d := \inf_{y\in K} ||x - y||$.

We claim that $\{y_n\}$ is Cauchy. If so, then y_n converges to some $y \in H$, and therefore $y \in K$; since $|| \cdot ||$ is continuous, we have

$$||x - y|| = \lim_n ||x - y_n|| = d,$$

and so the inf is achieved.

Proof of Cauchy:

By parallelogram law, with $y_n - x$ and $y_m - x$ as sides,

$$2(||y_n - x||^2 + ||y_m - x||^2) = ||y_n - y_m||^2 + ||y_n + y_m - 2x||^2$$

Re-write:

$$2(||y_n - x||^2 + ||y_m - x||^2) = ||y_n - y_m||^2 + 4||(y_n + y_m)/2 - x||^2$$

Since K is convex, $(y_n+y_m)/2 \in K$ and so $||(y_n+y_m)/2-x||^2 \geq d^2$. So, for large n, m,

$$\boxed{(\approx 4d^2) = ||y_n - y_m||^2 + (\geq 4d^2)}$$

Thus, $||y_n - y_m||^2$ is small.

With a bit more rigor:

Re-write:

$$||y_n - y_m||^2 = 2(||y_n - x||^2 + ||y_m - x||^2) - 4||(y_n + y_m)/2 - x||^2$$

$$\leq 2(||y_n - x||^2 + ||y_m - x||^2) - 4d^2,$$

the latter inequality since $(1/2)(y_n+y_m) \in K$, owing to the convexity of K.

Given $\epsilon > 0$, for large n, m,

$$2(||y_n - x||^2 + ||y_m - x||^2) - 4d^2 \leq 2(d^2 + \epsilon + d^2 + \epsilon) - 4d^2 = 4\epsilon.$$

Thus, $\{y_n\}$ is Cauchy.

Uniqueness: Suppose that $y, y' \in K$ are distinct points that achieve the inf:

$$||x - y|| = d = ||x - y'||$$

Then by parallelogram law with sides $x - y$ and $x - y'$,

$$4d^2 = 2(||x - y||^2 + ||x - y'||^2) = ||2x - y - y'||^2 + ||y - y'||^2$$

$$= 4||x - (y + y')/2||^2 + ||y - y'^2|| > 4||x - (y + y')/2||^2,$$

and so the distance from x to the midpoint $(y + y')/2$ is strictly smaller than d, a contradiction. □

Q: Counterexample for closed, but not convex set in Hilbert space? (without convexity, uniquenss is false).

Defn: Let H be a Hilbert space. For any set $E \subseteq H$, the *orthogonal complement* is defined

$$E^\perp = \{y \in H : \forall x \in E, x \perp y\}$$

Prop: $E^\perp$ is a closed subspace of H.

Proof: By bilinearity, $E^\perp$ is a subspace. It is closed because $\langle x, y\rangle$ is continuous: if $y_n \in E^\perp$ and $y_n \to y$ in H, then for all $x \in E$, we have $\langle x, y_n\rangle = 0$ and so

$$\langle x, y\rangle = \langle x, \lim_n y_n\rangle = \lim_n \langle x, y_n\rangle = 0. \square$$

Theorem: Let M be a closed subpsace of a Hilbert space H. Then $H = M \oplus M^{\perp}$, i.e., each $x \in H$ is uniquely expressible as $y + z$ where $y \in M, z \in M^{\perp}$. In fact, y, z are the unique points in $M, M^{\perp}$ closest to x.

Proof: By Theorem (closest point), there is a unique $y \in M$ closest to x (of course, if $x \in M$, then $y = x$). Let $z = x - y$.

We will show that $z \in M^{\perp}$, i.e., for all $v \in M$ $< z, v >= 0$.

We claim for all $t \in \mathbb{R}$

$$||z|| \leq ||z + tv||$$

To see this, observe that $||z||$ is the distance from x to y (the unique closest point on M to x), and $||z+tv|| = ||x-y-tv||$ is the distance from x to $y + tv \in M$.

Thus, $f(t) = ||z + tv||^2$ achieves a minimum at $t = 0$. But

$$f(t) = ||z+tv||^2 =< z+tv, z+tv >=< z, z > +2t\Re < z, v > +t^2 < v, v >$$

which is a differentiable function on $\mathbb{R}$. So, $t = 0$ is a critical point and thus $f'(0) = 0$. But $f'(t) = 2\Re < z, v > +2t < v, v >$ and so $0 = f'(0) = 2\Re < z, v >$ and so $\Re < z, v >= 0$. Applying the same argument to $||z + itv||^2$, we get $\Im < z, v >= 0$.

So, $x = y + z, y \in M, z \in M^{\perp}$.

Uniqueness is easy: If $x = y + z = y' + z'$, with $y, y' \in M$, $z, z' \in M^{\perp}$, then $y - y' = z' - z \in M \cap M^{\perp}$ and so $y - y' \perp y - y'$ and so $y - y' = 0$ and thus also $z - z' = 0$. $\square$

For any subpsaces is $(M^{\perp})^{\perp} = M$? Need closed subspace and is then true.

Proof if H were finite dimensional:

$dim(M^{\perp}) = dim(H) - dim(M)$ and so

$$dim(M^{\perp})^{\perp} = dim(H) - dim(M^{\perp})$$
$$= dim(H) - (dim(H)) - dim(M) = dim(M).$$

But clearly $(M^{\perp})^{\perp} \supset M$, and then since they have the same dimension, $(M^{\perp})^{\perp} = M$. $\square$

Prop: Let H be a Hilbert space.

1. For any set $E \subseteq H$, $(E^{\perp})^{\perp}$ is the smallest closed subspace that contains E.

2. If M is a closed subspace, then $(M^{\perp})^{\perp} = M$.

Proof: 2 follows from 1 since M is the smallest closed subspace that contains M.

Proof of 1: Will use $A \subseteq B$ implies $A^{\perp} \supseteq B^{\perp}$.

$(E^{\perp})^{\perp}$ is a closed subspace that contains E. $(E^{\perp})^{\perp} \supseteq E$

Since an arbitrary intersection of closed sets is closed and an arbitrary intersection of subspaces is a subspace, the smallest closed subspace that contains a given set is the intersection of all closed subspaces that contains the given set.

Let N be the smallest closed subspace that contains E. Then $N \cap (E^{\perp})^{\perp} = N$ and so

$$E \subset N \subset (E^{\perp})^{\perp}$$

Suppose $(E^{\perp})^{\perp} \neq N$. Then there exists $x \in (E^{\perp})^{\perp} \setminus N$.

Since $N \oplus N^{\perp} = H$, we can write $x = y + z, y \in N, z \in N^{\perp}$.

Since $z \in N^{\perp}$, $\underline{\langle y, z\rangle = 0}$.