Recall C-S

$||x|| = \sqrt{\langle x, x \rangle}$

Lecture 15:

The norm induced by an inner product satisfies triangle inequality (and thus is a norm).

Proof: We already checked positivity and homogeneity. It remains to check triangle inequality.

Fun fact: $w + \overline{w} = 2\Re(w)$ because

$$(x + yi) + (x - yi) = 2x.$$

By C-S inequality,

$$||x+y||^2 = \langle x+y, x+y \rangle = ||x||^2 + ||y||^2 + 2\Re\langle x, y \rangle \le ||x||^2 + ||y||^2 + 2|\langle x, y \rangle|$$

$$\le ||x||^2 + ||y||^2 + 2||x||\ ||y|| = (||x|| + ||y||)^2 \quad \square$$

As in $\mathbb{R}^n$, since $0 \le \frac{<x,y>}{||x||||y||} \le 1$, the inner product can be interpreted as the cosine of the angle between x and y.

Defn: x and y are *orthogonal*, denoted $x \perp y$, if $< x, y >= 0$.

Pythagorean Theorem holds in an inner product space: If $x_1, \ldots, x_n \in H$ are mutually orthogonal, then

$$||\sum_1^n x_j||^2 = \sum_1^n ||x_j||^2$$

Proof:

$$||\sum_1^n x_j||^2 = \langle \sum_1^n x_j, \sum_1^n x_j \rangle$$

$$= \sum_1^n < x_j, x_j > + 2\Re \sum_{i<j} \langle x_i, x_j \rangle = \sum_1^n ||x_j||^2$$

since the cross-terms cancel out. $\square$

Parallelogram Law holds in an inner product space:

$$2||x||^2 + 2||y||^2 = ||x+y||^2 + ||x-y||^2$$

Proof:

$$||x+y||^2 = ||x||^2 + ||y||^2 + 2\Re < x, y >$$
$$||x-y||^2 = ||x||^2 + ||y||^2 - 2\Re < x, y >$$

Add the preceding identities. □

Main examples of inner product space:

$$\mathbb{R}^n, \ \textit{with dot product } \langle x, y \rangle = \sum_i x_i y_i$$

$$\mathbb{C}^n, \ \textit{with conjugate dot product} \langle x, y \rangle = \sum_i x_i \overline{y_i}$$

$$L^2, \ \langle f, g \rangle = \int f \bar{g} d\mu$$

Exists by Holder inequality, since $p = 2$ is its own conjugate exponent.

Positivity: $\int f\bar{f}d\mu = \int |f|^2 d\mu = 0$ implies $f = 0$ a.e.

Symmetry:

$$\langle g, f \rangle = \int g\bar{f}d\mu = \overline{\int f\bar{g}d\mu} = \overline{\langle f, g \rangle}$$

Bilinearity:

$$\langle f+g, h \rangle = \int (f+g)\bar{h}d\mu = \int f\bar{h}d\mu + \int g\bar{h}d\mu = \langle f, h \rangle + \langle g, h \rangle.$$

Special case $\ell^2 = L^2(\mathbb{N})$.

Defn: A *Hilbert space* is a complete inner product space (with the metric induced by the norm).

Example: L^2 is complete and therefore as an inner product space is a Hilbert space.

Example: c_c is a subspace of ℓ^2 and thus can be viewed as an inner product space. But it it not closed, and therefore not complete, in the ℓ^2 norm, e.g., $x = (1, 1/2, 1/3, \ldots)$ is in ℓ^2 and its truncations

$$x_n = (1, 1/2, \ldots, 1/n, 0, 0, 0, \ldots)$$

are in c_c and $x_n \to x$ in ℓ^2, but $x \notin c_c$.

Note: a necessary condition for an NVS to be an inner product space is that it obeys the parallellogram law.

Prop: L^p obeys the paralleogram law iff $p = 2$.

Cor: L^p is a Hibert space iff $p = 2$.

Proof for special case ℓ^p.

"If:" we already know that ℓ^2 is an inner product space and thus satisfies the parallelogram law.

"Only If:" Let e_1, e_2 be the standard basis vectors

$$e_1 = (1, 0, 0, 0, \ldots), e_2 = (0, 1, 0, 0, \ldots),$$

The two sides of the parallelogram law for these vectors in ℓ^p are:

$$||e_1 + e_2||_p^2 + ||e_1 - e_2||_p^2 = 22^{2/p}$$

$\leftarrow \langle 1\,1\,0\,0 \ldots \rangle$

and

$$2(||e_1||_p^2 + ||e_2||_p^2) = 4$$

$(2^{1/p})^2$

They are equal iff $4 = 22^{2/p}$, equivalently $p = 2$.

So, if ℓ^p is a Hilbert space, then $p = 2$. $\square$

In fact:

An NVS is an inner product space iff its norm satisfies parallelogram law.

Proof: "only if" is clear.

"if": formula for inner product from norm; in the real case, this is

$$\langle x, y\rangle = (1/4)(||x+y||^2 - ||x-y||^2)$$

in the complex case, this is

$$\langle x, y\rangle = (1/4)(||x+y||^2 - ||x-y||^2 + i||x+iv||^2 - i||x-iv||^2)$$

One can show assuming the parallelogram law, these really are inner products (maybe in HW4).

Corollary: A Banach space is a Hilbert space iff its norm satisfies parallelogram law.

average ~~average~~ ~~average~~

Defn: For a vector space X, a *convex combination* of $x, y \in X$ is a point of the form $tx + (1-t)y$ such that $t \in [0, 1]$. A subset S of a NVS X is *convex* if whenever $x, y \in S$ then every convex combination of x, y is in S. ~~lines~~ lines

Example: Any subspace of a vector space is convex.

Theorem: Let K be a nonempty closed convex subset of a Hilbert space H. Let $x \in H \setminus K$. Then $\inf_{y\in K} ||x-y||$ is achieved uniquely.

HW4: This is false (both existence and uniqueness) in Banach spaces, even when K is a closed subspace.

Proof: Let $\{y_n\}$ be a sequence in K such that $||x - y_n||$ converges to $d := \inf$.

We claim that $\{y_n\}$ is Cauchy. If so, then it converges to some $y \in K$ and since $\langle\cdot,\cdot\rangle$ is continuous (HW4), we have $||x-y|| = d$ and so the inf is achieved.

because K is closed

Proof of Cauchy:

By parallelogram law, with $y_n - x$ and $y_m - x$ as sides,

$$2(||y_n - x||^2 + ||y_m - x||^2) = ||y_n - y_m||^2 + ||y_n + y_m - 2x||^2$$

$$\geq 2d^2 + 2d^2 \qquad 4\left\|\frac{y_n+y_m}{2} - x\right\|^2 \geq 4d^2$$

We can re-write the preceding inequality as

$$||y_n - y_m||^2 = 2(||y_n - x||^2 + ||y_m - x||^2) - 4||(y_n + y_m)/2 - x||^2$$
$$\leq 2(||y_n - x||^2 + ||y_m - x||^2) - 4d^2,$$

the latter inequality since $(1/2)(y_n + y_m) \in K$, owing to the convexity of K.

Given $\epsilon > 0$, for large n, m,

$$2(||y_n - x||^2 + ||y_m - x||^2) - 4d^2 \leq 2(d^2 + \epsilon + d^2 + \epsilon) - 4d^2 = 4\epsilon.$$

Thus, $\{y_n\}$ is Cauchy.

Uniqueness: Suppose that $y, y' \in K$ are distinct points that achieve the inf:

$$||x - y|| = d = ||x - y'||$$

Then by parallelogram law with sides $x - y$ and $x - y'$,

$$4d^2 = 2(||x - y||^2 + ||x - y'||^2) = ||2x - y - y'||^2 + ||y - y'^2||$$
$$= 4||x - (y + y')/2||^2 + ||y - y'^2|| > 4||x - (y + y')/2||^2,$$

and so the distance from x to the midpoint $(y + y')/2$ is strictly smaller than d, a contradiction. □

Q: Counterexample for closed, but not convex set in Hilbert space?

d^2