Lecture 14:

We often write $X = Y$ when we mean that X and Y are isometrically isomorphic. This is the strongest notion of "sameness."

We were proving:

Theorem (Folland Thm 6.15) Let $1 < p < \infty, 1/p + 1/q = 1$. Then, $(L^p)^* = L^q$. More precisely, $(L^p)^*$ and L^q are isometrically isomorphic.

Recall that

$$L_g : L^p \to \mathbb{R}, L_g(f) = \int fg d\mu$$

is a BLF, and and we were showing that the map

$$L^q \mapsto (L^p)^*, \quad g \mapsto L_g$$

is an isometric isomorphism.

We showed all but surjectivity.

For surjectivity, we showed given, $\phi \in (L^p)^*$, there exists $g \in L^1$ (a R-N derivative) s.t. $L_g = \phi$.

It remains to show $g \in L^q$. .

Proof: (Folland, Theorem 6.14):

Let g_n be simple functions which *pointwise* converge to g s.t. $|g_n| \leq |g|$ (approximate positive and negative parts).

Since g_n are simple, $g_n \in L^q$.

Replace g_n by

Let

$$f_n = \frac{|g_n|^{q/p}\mathrm{sgn}(g)}{||g_n||_q^{q/p}}.$$

$g_n \chi_{\{x : g \neq 0\}}$

Then

$$|f_n|^p = \frac{|g_n|^q}{||g_n||_q^q}$$

Egoroff

So

$$||f_n||_p^p = \int \frac{|g_n|^q}{||g_n||_q^q} = 1$$

So, $||f_n||_p = 1$. Also,

$$|f_n g_n| = \frac{|g_n|^{q/p+1}}{||g_n||_q^{q/p}} = \frac{|g_n|^q}{||g_n||_q^{q-1}}$$

And so

$$\int |f_n g_n| d\mu = \int \frac{|g_n|^q}{||g_n||_q^{q-1}} d\mu = ||g_n||_q$$

Since $g_n \to g$, by Fatou,

$$||g||_q \leq \liminf ||g_n||_q = \liminf \int |f_n g_n| d\mu \leq \liminf \int |f_n g| d\mu$$

=?

$$= \liminf \int f_n g d\mu = \liminf \phi(f_n) \leq \liminf |\phi(f_n)|$$

$$\leq \liminf ||\phi|| \; ||f_n||_p = ||\phi|| < \infty$$

So, $g \in L^q$. $\square$

It also turns out that $(L^1)^* = L^\infty$ but $(L^\infty)^* \neq L^1$. Of course, L^1 isometrically embeds in its doubke dual $(L^\infty)^*$.

Hilbert spaces

Defn: *Real Inner Product space* Let X be a real vector space. An *inner product* on X is a function $X \times X \to \mathbb{R}$, $(x, y) \mapsto \langle x, y \rangle$

Positivity $\langle x, x \rangle \geq 0$ and $= 0$ iff $x = 0$

Symmetry $\langle x, y \rangle = \langle y, x \rangle$

Bilinearity: for fixed x, $\langle x, \cdot \rangle$ and $\langle \cdot, x \rangle$ are linear.

Given symmetry, need only one bilinearity equation.

$(X, \langle x, y \rangle)$ is called a *real inner product space.*

Defn: *Complex Inner Product space* Let X be a complex vector space. An *inner product* on X is a function $X \times X \to \mathbb{C}$, $(x, y) \mapsto \langle x, y \rangle$

Positivity: same as above.

Skew-Symmetry: $\langle x, y \rangle = \overline{\langle y, x \rangle}$

Sesqui-linearity: for fixed x, $\langle \cdot, x \rangle$ is linear and $\langle x, \cdot \rangle$ is skew-linear, i.e.,

$\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$ and

$\langle x, \lambda y \rangle = \overline{\lambda} \langle x, y \rangle$ Given skew-symmetry, need only one bilinearity equation.

$(X, \langle x, y \rangle)$ is called a *complex inner product space.*

Defn: *Induced Norm*: $||x|| = \sqrt{\langle x, x \rangle}$

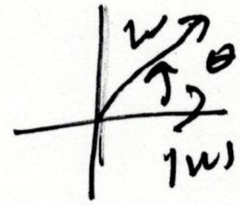
Verify that the induced norm is indeed a norm:

Positivity: follows from positivity of the inner product.

Homogeneity: $||ax|| = \sqrt{< ax, ax >} = \sqrt{a\overline{a}} < x, x >= |a| ||x||$ by Sesqui-linearity of the inner product.

Triangle Inequality:

First prove Cauchy-Schwarz:



$$|\langle x, y\rangle| \le ||x||\ ||y||$$

with equality iff x and y are linearly dependent.

Complex number fact: if $w \neq 0$, then $\alpha = \mathrm{sgn}(w) = w/|w| = e^{i\theta}$, where θ is the angle from pos. x-axis to w. So,

$$w = \alpha|w|, \quad \overline{w} = \overline{\alpha}|w| = \alpha^{-1}|w|$$

Proof of C-S for $\mathbb{C}$: (you have probably already seen the proof for $\mathbb{R}^n$).

If $\langle x, y\rangle = 0$, we are done.

So, assume $\langle x, y\rangle \neq 0$.

Let $\alpha = \mathrm{sgn}\langle x, y\rangle$. So,

$$\langle x, y\rangle = \alpha|\langle x, y\rangle|, \quad \langle y, x\rangle = \alpha^{-1}|\langle x, y\rangle|,$$

Let $z = \alpha y$.

Then

$$\langle z, x\rangle = \alpha\langle y, x\rangle = \alpha\alpha^{-1}|\langle x, y\rangle| = |\langle x, y\rangle|.$$

Similarly,

$$\langle x, z\rangle = \alpha^{-1}\langle x, y\rangle = \alpha^{-1}\alpha|\langle x, y\rangle| = |\langle x, y\rangle|.$$

For real t,

$$0 \le ||\cancel{x-tz}, x - tz||^2 = \langle x - tz, x - tz\rangle = ||x||^2 - 2t|\langle x, y\rangle| + t^2||y||^2$$

a quadratic with minimum achieved at $t = \frac{|\langle x,y\rangle|}{||y||^2}$.

Plugging in t, we get

$$0 \le ||x - tz||^2 = ||x||^2 - \frac{|\langle x, y\rangle|^2}{||y||^2}$$

"only if:" equality ⇒ $0 = ||x - tz||^2 \Rightarrow$
$x - tz = 0 \Rightarrow$
$x - \alpha t y = 0$

Transposing we get

$$|\langle x, y\rangle|^2 \leq ||x||^2||y||^2.$$

with equality iff $x - tz = x - \alpha ty = 0$, equivalently x and y linearly dependent. $\square$

"if" is obvious: $x = ry \Rightarrow |\langle x, y\rangle| = |r|\,||y||^2$ and $||x|| \cdot ||y|| = |r|\,||y|| \cdot ||y||$

The norm induced by an inner product satisfies triangle inequality (and thus is a norm).

Proof:

Complex number fact: $w + \overline{w} = 2\Re(w)$ because

$$(x + yi) + (x - yi) = 2x.$$

By C-S inequality,

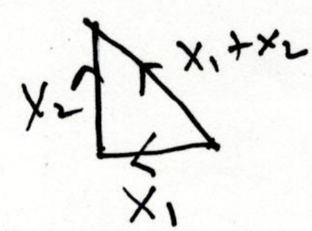
$$||x + y||^2 = ||x||^2 + ||y||^2 + 2\Re\langle x, y\rangle \leq ||x||^2 + ||y||^2 + 2|\Re\langle x, y\rangle|$$
$$\leq ||x||^2 + ||y||^2 + 2||x||\;||y|| = (||x|| + ||y||)^2 \quad \square$$

As in $\mathbb{R}^n$, since $0 \leq \frac{|<x,y>|}{||x||||y||} \leq 1$, the inner product can be interpreted as the cosine of the angle between x and y.

Defn: x and y are *orthogonal*, denoted $x \perp y$, if $< x, y >= 0$.

Pythagorean Theorem holds in an inner product space: If $x_1, \ldots, x_n \in H$ are mutually orthogonal, then

$$||\sum_1^n x_j||^2 = \sum_1^n ||x_j||^2$$



Proof:

$$||\sum_1^n x_j||^2 = \langle\sum_1^n x_j, \sum_1^n x_j\rangle$$

$$= \sum_1^n < x_j, x_j > +2\Re\sum_{i<j}\langle x_i, x_j\rangle = \sum_1^n ||x_j||^2$$