HW5, Due Friday, March 29, 11AM

1. Let $X$ and $Y$ be NVS. Show that the product topology of $X \times Y$, with $X$ and $Y$ given their norm topology, is the same as the topology given by the product norm: $||(x, y)|| = ||x|| + ||y||$.

Proof: a set is open in the product topology iff it is a union of sets $U \times V$ where $U$ is open in $X$ and $V$ is open in $Y$ iff it is union of set $B_\gamma(x) \times B_\delta(y)$.

$B_\gamma((x, y)) = \bigcup_{\delta \leq \gamma} B_\delta(x) \times B_{\gamma-\delta}(y)$

Given $(x, y)$ and $\gamma > 0$,

$$(x, y) \in B_{\gamma/2}(x) \times B_{\gamma/2}(y) \subset B_\gamma((x, y))$$

Given $(x, y)$, $\epsilon, \delta > 0$, let $\gamma = \min(\epsilon, \delta)$,

$$(x, y) \in B_\gamma((x, y)) \subset B_\delta(x) \times B_\epsilon(y) \quad \Box$$

2. In a topological space, the closure of a subset $A$ is the intersection of all closed sets that contain $A$. And the interior of a subset $A$ is the union of all open sets that are contained in $A$. Show the following.

(a) The closure of a set is closed, and a set is closed iff it equals its closure.

(b) The closure of $A$ is the set of all $x$ such that every neighbourhood of $x$ intersects $A$.

(c) The interior of a set is open, and the set is open iff it equals its interior.

(d) The interior of $A$ is the set of all $x$ such that there exists a neighbourhood of $x$ that is contained in $A$.

Solution:

a.

i. The closure of a set is the intersection of closed sets and thus is closed.

ii. If a set $A$ is closed then it is the intersection of all closed sets containing $A$, and thus $A$ equals its closure. If a set equals its closure, then by virtue of i, it is closed.

b. Suppose $x \in \overline{A}$. If some nbhd. $U$ of $x$ is disjoint from $A$, then $U^c$ is a closed set containing $A$ and therefore containing the closure of $A$, contradicting $x \in \overline{A}$.

Suppose that every neighbourhood of $x$ intersects $A$. If $x \notin \overline{A}$, then $\overline{A}^c$ is a nbhd. of $x$ that is disjoint from $A$, a contradiction. Thus, $x \in \overline{A}$.

Solutions to c and d are obtained by complementing a and b, using the fact that for any set $A$ the interior of $A$ is $(\overline{A})^c$.
3. Show that a set is a countable union of nowhere dense sets iff its complement contains the intersection of a countable collection of dense open sets.

Proof: “Only if:” $A = \bigcup E_i$ where each $E_i$ is nowhere dense. Then $A^c \supseteq \bigcap_i (E_i)^c$. Each $(E_i)^c$ is clearly open. Since $E_i$ is nowhere dense, $E_i^c$ contains no open set and thus every nonempty open set intersects $(E_i)^c$. Thus, $(E_i)^c$ is dense.

“If” Suppose $B \supseteq \bigcap_i U_i$ where each $U_i$ is open and dense. Then $B^c \subseteq \bigcup_i (U_i)^c$ and so $B^c = \bigcup_i (U_i)^c \cap B^c$ and since $(U_i)^c \cap B^c$ is contained in $(U_i)^c$, a closed nowhere dense set, and since any subset of a nowhere dense is nowhere dense, $B^c$ is a countable union of nowhere dense sets. □

4. Which of the following NVS are separable?: $C([0, 1]), C^1([0, 1]), C_0(\mathbb{R}), C_c(\mathbb{R}), L^\infty(\mathbb{R}, \mu)$ where $\mu$ is Lebesgue measure? (for each of these spaces the norm is the sup norm except for $C^1([0, 1])$ whose norm is $\|f\|_{C^1} = \|f\|_{\text{sup}} + \|f'\|_{\text{sup}}$ and for $L^\infty(\mathbb{R}, \mu)$ whose norm is the essential sup norm). For each that is separable, exhibit a countable dense subset (and at least give a rough argument for why the subset is dense). For each that is not separable, give a complete argument.

Solution: $C([0, 1])$, with sup norm, is separable by Stone-Weirstrass with polynomials with rational coefficients as countable dense set.

$C^1([0, 1])$ has norm $\|f\|_{C^1} = \|f\|_{\text{sup}} + \|f'\|_{\text{sup}}$.

Claim: $C^1([0, 1])$ is separable with polynomials with rational coefficients as countable dense set.

Proof: Given $f \in C^1([0, 1])$ and $\epsilon > 0$, find by Stone-Weirstrass, a polynomial $p(x)$ s.t. $\|f' - p\|_{\text{sup}} < \epsilon$. Let $P(x) = \int_0^x p(t)dt$. Then for all $x \in [0, 1],$

$$|f(x) - f(0) - P(x)| = |\int_0^x f'(t)dt - \int_0^x p(t)dt| \leq \int_0^x |f'(t) - p(t)|dt \leq \epsilon.$$ 

Let $Q(x) = f(0) + P(x)$. Then $\|f - Q\|_{\text{sup}} \leq \epsilon$ and $\|f' - Q'\|_{\text{sup}} = \|f' - p\|_{\text{sup}} \leq \epsilon$.

Now, approximate the polynomial $Q(x)$ with a polynomial with rational coefficients. □

$C_0(\mathbb{R})$ and $C_c(\mathbb{R})$, with sup norm, are separable with countable dense set $\bigcup_{n=1}^\infty D_n$ where $D_n$ is the set of all functions of the form

$$\begin{align*}
p(x) & \quad -n \leq x \leq n \\
q(x) & \quad -n - 1 \leq x \leq -n \\
r(x) & \quad n \leq x \leq n + 1 \\
0 & \quad |x| > n + 1
\end{align*}$$
where \( p(x) \) is a polynomial with rational coefficients, \( q(x) \) is the function whose graph is the line determined \((-n - 1, 0)\) and \((-n, p(-n))\), \( r(x) \) is the function whose graph is the line determined by \((n, p(n))\) and \((n + 1, 0)\).

\( L^\infty(\mathbb{R}, \mu) \), with sup norm, is not separable. As in the argument given in class that \( \ell^\infty \) is not separable, it suffices to find an uncountable collection of elements of \( L^\infty(\mathbb{R}, \mu) \) whose pairwise distances are all equal to 1. Such a collection is

\[
\{ \sum a_n \chi_{(n,n+1)} : a_n \in \{0, 1\} \}
\]

5. Let \( X, Y \) be Banach spaces. Show that the collection of surjective BLTs from \( X \) onto \( Y \) is open in the space \( L(X, Y) \) of BLTs, with the operator norm topology on \( L(X, Y) \).

Solution:

If there is no surjective element of \( L(X, Y) \), we are done already.

Let \( T \) be a surjective BLT from \( X \) onto \( Y \). By the open mapping theorem, there exists \( r > 0 \) s.t. \( B_1(0) \subset T(B_r(0)) \).

Let \( S \in L(X, Y) \) such that \( \|S - T\| < \frac{1}{2r} \). We claim that \( S \) is surjective.

We will show that any \( y \in Y \) is in the image of \( S \). We can assume that \( y \neq 0 \) and \( \|y\| \leq 1 \) by replacing \( y \) with \( \frac{y}{\|y\|} \). We constuct, by induction, an absolutely convergent sequence \( x_i \) such that \( \|S(\sum_{i=0}^N x_i) - y\| \leq \frac{1}{2i+1} \) for all \( N \).

Observe that \( y \in \overline{B}_1(0) \subset T(\overline{B}_r(0)) \), so there is some \( x_0 \in \overline{B}_r(0) \) such that \( y = T(x_0) \) and

\[
\|S(x_0) - y\| = \|S(x_0) - T(x_0)\| \leq \|S - T\| \|x_0\| \leq \frac{1}{2r} \cdot \frac{1}{2} = \frac{1}{2}.
\]

From this we get that \( (S(x_0) - y) \in \overline{B}_{1/2}(0) \), so again using \( \overline{B}_1(0) \subset T(\overline{B}_r(0)) \), we get some \( x_1 \in \overline{B}_{r/2}(0) \) such that \( T(x_1) = (y - S(x_0)) \). So

\[
\|S(x_0 + x_1) - y\| = \|S(x_1) - (y - S(x_0))\| = \|S(x_1) - T(x_1)\| \leq \|S - T\| \|x_1\| \leq \frac{1}{2} \cdot \frac{r}{2} = \frac{1}{4}.
\]

Continuing the process by induction, we get a sequence \( (x_n) \) in \( X \) such that \( \|x_n\| \leq r/2^n \) and

\[
\left\| \sum_{i=0}^n x_i - y \right\| \leq \frac{1}{2^{n+1}}.
\]

Indeed, at step \( n \), we get \( x_{n+1} \in B_{r/2^{n+1}}(0) \) such that \( T(x_{n+1}) = y - S(\sum_{i=0}^n x_i) \). So,

\[
\|S(\sum_{i=0}^{n+1} x_i) - y\| = \|S(x_{n+1}) - (y - S(\sum_{i=0}^n x_i))\|
\]

\[
= \|S(x_{n+1}) - T(x_{n+1})\| \leq \|S - T\| \|x_{n+1}\| \leq \frac{1}{2r} \cdot \frac{r}{2^{n+1}} = \frac{1}{2^{n+2}}.
\]

By completeness of \( X \), absolute convergence of \( x_n \), and continuity of \( S \), \( \sum_{n=1}^\infty x_n \) converges to some \( x \in X \) such that \( S(x) = y \).
6. Let \( H \) be the space of absolutely continuous functions \( f : [0, 1] \to \mathbb{C} \) such that \( f(0) = f(1) = 0 \), and \( f' \in L^2([0, 1]) \).

(a) Show that \( H \) is a Hilbert space with inner product 
\[
\langle f, g \rangle = \int_{[0,1]} f'(x) \overline{g'(x)} \, dx.
\]

(b) Consider the evaluation map \( \text{ev}_a(f) = f(a) \). Find the unique \( f_a \in H \) such that for all \( f \in H \) we have 
\[
f(a) = \text{ev}_a(f) = \langle f, f_a \rangle.
\]

Hint: Consider a piecewise linear function on \([0, 1]\).

Solution:

a. It is straightforward to check that the inner product really is an inner product space. For completeness, let \( f_n \) be a Cauchy sequence in this space. This means that \( f_n' \) is Cauchy in \( L^2 \) and since \( L^2 \) is complete, \( f_n' \) converges to some \( g \) in \( L^2 \). Let \( f(x) = \int_0^x g(t) \, dt \). Then \( f'(x) = g(x) \) a.e. Clearly, \( f(0) = 0 \). We claim that \( f(1) = 0 \). To see this, observe that by the Holder inequality,
\[
||f'||_1 - ||f'_n||_1 \leq ||f' - f'_n||_1 \leq ||f' - f'_n||_2
\]
Since each \( ||f'_n||_1 = f_n(1) - f_n(0) = 0 \), it follows that \( f(1) - f(0) = ||f'||_1 = 0 \) and so \( f(1) = 0 \).
Thus, \( f \) is absolutely continuous, \( f(1) = f(0) = 0 \), and \( f' \in L^2 \), we have \( f \in H \) and \( f_n \) converges to \( f \) in \( H \).

b. the solution is for \( f_a \) to be a triangular function 
\[
f_a(t) = \begin{cases} 
  x & \text{if } 0 \leq x \leq a \\
  a - \frac{a}{1-a}(x-a) & \text{for } a \leq x \leq 1
\end{cases}
\]

Then
\[
\langle f, f_a \rangle = \int_0^a f'(t) \, dt - \frac{a}{1-a} \int_a^0 f'(t) \, dt = (f(a) - f(0)) - \frac{a}{1-a}(f(1) - f(a)) = f(a)(1 + \frac{a}{1-a}) = f(a).
\]