HW5, Due Friday, March 29, 11AM

1. Let X and Y be NVS. Show that the product topology of $X \times Y$, with X and Y given their norm topology, is the same as the topology given by the product norm: $||(x,y)|| = ||x|| + ||y||$.

   Proof: a set is open in the product topology iff it is a union of sets $U \times V$ where U is open in X and V is open in Y iif it is union of set $B_\epsilon(x) \times B_\delta(y)$.

$$B_\gamma((x,y)) = \cup_{\delta \leq \gamma} B_\delta(x) \times B_{\gamma-\delta}(y)$$

   Given (x,y) and $\gamma > 0$,

$$(x,y) \in B_{\gamma/2}(x) \times B_{\gamma/2}(y) \subset B_\gamma((x,y))$$

   Given (x,y), $\epsilon, \delta > 0$, let $\gamma = \min(\epsilon, \delta)$,

$$(x,y) \in B_\gamma((x,y)) \subset B_\delta(x) \times B_\epsilon(y) \quad \Box$$

2. In a topological space, the *closure* of a subset A is the intersection of all closed sets that contain A. And the *interior* of a subset A is the union of all open sets that are contained in A. Show the following.

   (a) The closure of a set is closed, and a set is closed iff it equals its closure.

   (b) The closure of A is the set of all x such that every neighbourhood of x intersects A.

   (c) The interior of a set is open, and the set is open iff it equals its interior.

   (d) The interior of A is the set of all x such that there exists a neighbourhood of x that is contained in A.

   Solution:

   a.

   i. The closure of a set is the intersection of closed sets and thus is closed.

   ii. If a set A is closed then it is the intersection of all closed sets containing A, and thus A equals its closure. If a set equals its closure, then by virtue of i, it is closed.

   b. Suppose $x \in \overline{A}$. If some nbhd. U of x is disjoint from A, then U^c is a closed set containing A and therefore containing the closure of A, contradicting $x \in \overline{A}$.

   Suppose that every neighbourhood of x intersects A. If $x \notin \overline{A}$, then $\overline{A}^c$ is a nbhd. of x that is disjoint from A, a contradiction. Thus, $x \in \overline{A}$.

   Solutions to c and d are obtained by complementing a and b, using the fact that for any set A the interior of A is $(\overline{A})^c$.

3. Show that a set is a countable union of nowhere dense sets iff its complement contains the intersection of a countable collection of dense open sets.

   Proof: "Only if:" $A = \cup E_i$ where each E_i is nowhere dense. Then $A^c \supseteq \cap_i (\overline{E_i})^c$. Each $(\overline{E_i})^c$ is clearly open. Since E_i is nowhere dense, $\overline{E_i}$ contains no open set and thus every nonempty open set intersects $(\overline{E_i})^c$. Thus, $(\overline{E_i})^c$ is dense.

   "If" Suppose $B \supseteq \cap_i U_i$ where each U_i is open and dense. Then

$$B^c \subseteq \cup_i (U_i)^c$$

   and so

$$B^c = \cup_i (U_i)^c \cap B^c$$

   and since $(U_i)^c \cap B^c$ is contained in $(U_i)^c$, a closed nowhere dense set, and since any subset of a nowhere dense is nowhere dense, B^c is a countable union of nowhere dense sets. $\square$

4. Which of the following NVS are separable?: $C([0,1]), C^1([0,1]), C_0(\mathbb{R}), C_c(\mathbb{R}), L^\infty(\mathbb{R}, \mu)$ where μ is Lebesgue measure? (for each of these spaces the norm is the sup norm except for $C^1([0,1])$ whose norm is $||f||_{C^1} = ||f||_{\text{sup}} + ||f'||_{\text{sup}}$ and for $L^\infty(\mathbb{R}, \mu)$ whose norm is the essential sup norm). For each that is separable, exhibit a countable dense subset (and at least give a rough argument for why the subset is dense). For each that is not separable, give a complete argument.

   Solution: $C([0,1])$, with sup norm, is separable by Stone-Weirstrass with polynomials with rational coefficients as countable dense set.

   $C^1([0,1])$ has norm $||f||_{C^1} = ||f||_{\text{sup}} + ||f'||_{\text{sup}}$.

   Claim: $C^1([0,1])$ is separable with polynomials with rational coefficients as countable dense set.

   Proof: Given $f \in C^1([0,1])$ and $\epsilon > 0$, find by Stone-Weirstrass, a polynomial $p(x)$ s.t. $||f' - p||_{\text{sup}} < \epsilon$. Let $P(x) = \int_0^x p(t)dt$. Then for all $x \in [0,1]$,

$$|f(x) - f(0) - P(x)| = |\int_0^x f'(t)dt - \int_0^x p(t)dt| \leq \int_0^x |f'(t) - p(t)|dt \leq \epsilon.$$

   Let $Q(x) = f(0) + P(x)$. Then $||f - Q||_{\text{sup}} \leq \epsilon$ and $||f' - Q'||_{\text{sup}} = ||f' - p||_{\text{sup}} \leq \epsilon$.

   Now, approximate the polynomial $Q(x)$ with a polynomial with rational coefficients. $\square$

   $C_0(\mathbb{R})$ and $C_c(\mathbb{R})$, with sup norm, are separable with countable dense set $\cup_{n=1}^\infty D_n$ where D_n is the set of all functions of the form

$$\left\{ \begin{array}{cc} p(x) & -n \leq x \leq n \\ q(x) & -n-1 \leq x \leq -n \\ r(x) & n \leq x \leq n+1 \\ 0 & |x| > n+1 \end{array} \right\}$$

where $p(x)$ is a polynomial with rational coefficients, $q(x)$ is the function whose graph is the line determined $(-n-1,0)$ and $(-n,p(-n))$, $r(x)$ is the function whose graph is the line determined by $(n,p(n))$ and $(n+1,0)$.

$L^\infty(\mathbb{R},\mu)$, with sup norm, is not separable. As in the argument given in class that ℓ^∞ is not separable, it suffices to find an uncountable collection of elements of $L^\infty(\mathbb{R},\mu)$ whose pairwise distances are all equal to 1. Such a collection is

$$\{\sum a_n\chi_{(n,n+1)} : a_n \in \{0,1\}\}$$

5. Let X,Y be Banach spaces. Show that the collection of surjective BLTs from X onto Y is open in the space $L(X,Y)$ of BLTs, with the operator norm topology on $L(X,Y)$.

   Solution:

   If there is no surjective element of $L(X,Y)$, we are done already.

   Let T be a surjective BLT from X onto Y. By the open mapping theorem, there exists $r>0$ s.t. $\overline{B_1(0)} \subset T(\overline{B_r(0)})$.

   Let $S \in L(X,Y)$ such that $\|S-T\| < \frac{1}{2r}$. We claim that S is surjective.

   We will show that any $y \in Y$ is in the image of S. We can assume that $y \neq 0$ and $\|y\| \leq 1$ by replacing y with $\frac{y}{\|y\|}$. We constuct, by induction, an absolutely convergent sequence x_i such that $\|S(\sum_{i=0}^N x_i) - y\| \leq \frac{1}{2^{N+1}}$ for all N.

   Observe that $y \in \overline{B}_1(0) \subset T(\overline{B}_r(0))$, so there is some $x_0 \in \overline{B}_r(0)$ such that $y = T(x_0)$ and

$$\|S(x_0) - y\| = \|S(x_0) - T(x_0)\| \leq \|S-T\|\|x_0\| \leq \frac{1}{2r}r = \frac{1}{2}.$$

   From this we get that $(S(x_0)-y) \in \overline{B}_{1/2}(0)$, so again using $\overline{B}_1(0) \subset T(\overline{B}_r(0))$, we get some $x_1 \in B_{r/2}(0)$ such that $T(x_1) = (y - S(x_0))$. So

$$\|S(x_0+x_1)-y\| = \|S(x_1)-(y-S(x_0))\| = \|S(x_1)-T(x_1)\| \leq \|S-T\|\|x_1\| \leq \frac{1}{2r}\frac{r}{2} = \frac{1}{4}.$$

   Continuing the process by induction, we get a sequence (x_n) in X such that $||x_n|| \leq r/2^n$ and

$$\left\|S(\sum_{i=0}^n x_i) - y\right\| \leq \frac{1}{2^{n+1}}.$$

   Indeed, at step n, we get $x_{n+1} \in B_{r/2^{n+1}}(0)$ such that $T(x_{n+1}) = y - S(\sum_{i=0}^n x_i)$. So,

$$\|S(\sum_{i=0}^{n+1} x_i) - y\| = \|S(x_{n+1}) - (y - S(\sum_{i=0}^{n} x_i))\|$$

$$= \|S(x_{n+1}) - T(x_{n+1})\| \leq \|S-T\|\|x_{n+1}\| \leq \frac{1}{2r}\frac{r}{2^{n+1}} = \frac{1}{2^{n+2}}.$$

   By completeness of X, absolute convergence of x_n, and continuity of S, $\sum_{n=1}^\infty x_n$ converges to some $x \in X$ such that $S(x) = y$.

6. Let H be the space of absolutely continuous functions $f : [0,1] \to \mathbb{C}$ such that $f(0) = f(1) = 0$, and $f' \in L^2([0,1])$.

   (a) Show that H is a Hilbert space with inner product

   $$\langle f, g \rangle = \int_{[0,1]} f'(x)\overline{g'(x)}\, \mathrm{d}x.$$

   (b) Consider the evaluation map $\mathrm{ev}_a(f) = f(a)$. Find the unique $f_a \in H$ such that for all $f \in H$ we have

   $$f(a) = \mathrm{ev}_a(f) = \langle f, f_a \rangle .$$

   Hint: Consider a piecwise linear function on $[0,1]$.

Solution:

a. It is straightforward to check that the inner product really is an inner product space.

For completeness, let f_n be a Cauchy seqnence in this space. This means that f_n' is Cauchy in L^2 and since L^2 is complete, f_n' converges to some g in L^2. Let $f(x) = \int_0^x g(t)dt$. Then $f'(x) = g(x)$ a.e. Clearly, $f(0) = 0$. We claim that $f(1) = 0$. To see this, observe that by the Holder inequality,

$$|\ ||f'||_1 - ||f_n'||_1\ | \le ||f' - f_n'||_1 \le ||f' - f_n'||_2$$

Since each $||f_n'||_1 = f_n(1) - f_n(0) = 0$, it follows that $f(1) - f(0) = ||f'||_1 = 0$ and so $f(1) = 0$.

Thus, f is absolutely continuous, $f(1) = f(0) = 0$, and $f' \in L^2$, we have $f \in H$ and f_n converges to f in H.

b. the solution is for f_a to be a triangular function

$$f_a(t) = \begin{cases} x \text{ if } 0 \le x \le a \\ a - \frac{a}{1-a}(x-a) \text{ for } a \le x \le 1 \end{cases}$$

Then

$$\langle f, f_a \rangle = \int_0^a f'(t)\,\mathrm{d}t - \frac{a}{1-a}\int_a^0 f'(t) = (f(a) - f(0)) - \frac{a}{1-a}(f(1) - f(a)) = f(a)(1 + \frac{a}{1-a}) = f(a).$$