HW5, Due Friday, March 29, 11AM

1. Let X and Y be NVS. Show that the product topology of $X \times Y$, with X and Y given their norm topology, is the same as the topology given by the product norm: $||(x, y)|| = ||x|| + ||y||$.

2. In a topological space, the *closure* of a subset A is the intersection of all closed sets that contain A. And the *interior* of a subset A is the union of all open sets that are contained in A. Show the following.

   (a) The closure of a set is closed, and a set is closed iff it equals its closure.

   (b) The closure of A is the set of all x such that every neighbourhood of x intersects A.

   (c) The interior of a set is open, and the set is open iff it equals its interior.

   (d) The interior of A is the set of all x such that there exists a neighbourhood of x that is contained in A.

3. Show that a set is a countable union of nowhere dense sets iff its complement contains the intersection of a countable collection of dense open sets.

4. Which of the following NVS are separable?: $C([0,1]), C^1([0,1]), C_0(\mathbb{R}), C_c(\mathbb{R}), L^\infty(\mathbb{R}, \mu)$ where μ is Lebesgue measure? (for each of these spaces the norm is the sup norm except for $C^1([0,1])$ whose norm is $||f||_{C^1} = ||f||_{\text{sup}} + ||f'||_{\text{sup}}$ and for $L^\infty(\mathbb{R}, \mu)$ whose norm is the essential sup norm). For each that is separable, exhibit a countable dense subset (and at least give a rough argument for why the subset is dense). For each that is not separable, give a complete argument.

5. Let X, Y be Banach spaces. Show that the collection of surjective BLTs from X onto Y is open in the space $L(X, Y)$ of BLTs, with the operator norm topology on $L(X, Y)$.

6. Let H be the space of absolutely continuous functions $f : [0,1] \to \mathbb{C}$ such that $f(0) = f(1) = 0$, and $f' \in L^2([0,1])$.

   (a) Show that H is a Hilbert space with inner product
   $$\langle f, g \rangle = \int_{[0,1]} f'(x)\overline{g'(x)}\ \mathrm{d}x.$$

   (b) Consider the evaluation map $\mathrm{ev}_a(f) = f(a)$. Find the unique $f_a \in H$ such that for all $f \in H$ we have
   $$f(a) = \mathrm{ev}_a(f) = \langle f, f_a \rangle .$$

   Hint: Consider a piecwise linear function on $[0,1]$.