HW4, Due Friday, March 8, 11AM

1. (a) Show that the kernel of a BLF on a Banach space is closed.
   (b) Shat that the orthogonal complement of the kernel of a BLF on Hilbert space has dimension at most one.

   Solution:

   a. Let X denote the Banach space. Since any BLF is continuous, if $x_n \in \mathrm{Ker} f$ and $x_n \to x \in X$ then

   $$f(x) = \lim_{x_n \to x} f(x_n) = \lim_{x_n \to x} 0 = 0.$$

   b. By a, $W = \mathrm{Ker}\ f$ is a closed subspace of X, a Hilbert space, and thus has an orthogonal complement $W^\perp$.

   Let $x, x' \in W^\perp$ s.t. $f(x) = f(x')$. Then $x - x' \in Ker\ f$ and so $\langle x, x - x' \rangle = 0$ and $\langle x', x - x' \rangle = 0$ and so $\|x - x'\|^2 = \langle x - x', x - x' \rangle = 0$ and so $x = x'$. Thus, $f : W^\perp \to K$ is an injective linear map into K and thus a vector space isomorphism from $W^\perp$ onto its image which is either 0 or K and so $W^\perp$ has dimension 0 or 1.

2. Show that if $\|\cdot\|$ is a norm on a real vector space that satisfies the parallelogram law then

   $$\langle x, y \rangle := (1/4)(\|x + y\|^2 - \|x - y\|^2) \tag{1}$$

   is an inner product whose induced norm agrees with $\|\cdot\|$. And thus show that a real Banach space is a real Hilbert space iff its norm satisfies the parallelogram law.

   Hint: To prove bilinearity of $\langle x, y \rangle$,

   (a) first show

   $$\langle v + w, u \rangle = \langle v, u \rangle + \langle w, u \rangle$$

   (b) then use part (a) to show for all rational α

   $$\langle \alpha x, y \rangle = \alpha \langle x, y \rangle \tag{2}$$

   (c) then show (2) for all α.

   Solution:

   By substituting $y = x$ in the definition of the inner product, one sees that the norm induced by the inner product is indeed the original norm.

   To show that the alleged inner product really is an inner product, we check:

   Positivity: This follows immediately from properties of the norm.

   Symmetry: This is obvious.

   Bi-linearity: because of symmetry we need only check linearity in the first coordinate of the inner product.

3. Recall that for a complex Hilbert space H and $y \in H$, $y^*(x) = \langle x, y \rangle$ is a BLF.

   Show that $\langle y^*, z^* \rangle_{H^*} := \langle z, y \rangle_H$ is an inner product on H^*.

   Solution:

   Positivity:

$$\langle y^*, y^* \rangle_{H^*} := \langle y, y \rangle_H \geq 0 \text{ and } = 0 \text{ iff } y = 0 \text{ iff } y^* = 0$$

   Skew-symmetry:

$$\overline{\langle y^*, z^* \rangle_{H^*}} := \overline{\langle z, y \rangle_H} = \langle y, z \rangle_H = \langle z^*, y^* \rangle_{H^*}$$

   Sesqui-linearity: First observe that

$$ay^* + bz^* = (\overline{a}y + \overline{b}z)^*$$

   because for all x,

$$(ay^* + bz^*)(x) = ay^*(x) + bz^*(x) = a\langle x, y \rangle + b\langle x, z \rangle$$

$$= \langle x, \overline{a}y \rangle + \langle x, \overline{b}z \rangle = \langle x, \overline{a}y + \overline{b}z \rangle = (\overline{a}y + \overline{b}z)^*(x).$$

   So,

$$\langle ay^* + bz^*, w^* \rangle_{H^*} = \langle w, \overline{a}y + \overline{b}z \rangle_H$$

$$= a\langle w, y \rangle_H + b\langle w, z \rangle_H = a\langle y^*, w^* \rangle_{H^*} + b\langle z^*, w^* \rangle_{H^*}$$

   And similarly

$$\langle w^*, ay^* + bz^* \rangle_{H^*} = \langle w^*, (\overline{a}y + \overline{b}z)^* \rangle_{H^*} = \langle \overline{a}y + \overline{b}z, w \rangle_H$$

$$= \overline{a}\langle y, w \rangle_H + \overline{b}\langle z, w \rangle_H = \overline{a}\langle w^*, y^* \rangle_{H^*} + \overline{b}\langle w^*, z^* \rangle_{H^*}$$

4. Let V be a vector space and W be a subspace of V. Let V/W denote the set of equivalence classes of the relation $v_1 \sim v_2$ iff $v_1 - v_2 \in W$. So, the equivalence classes are of the form $[v] = v + W$. Define a vector space structure on V/W by

$$(v_1 + W) + (v_1 + W) = (v_1 + v_2) + W$$

   and

$$\lambda(v + W) = \lambda v + W.$$

   (a) Show that V/W is a well-defined vector space (just show that the vector addition and scalar multiplication operations on V/W are well-defined, identify the zero element of V/W amd the additive inverses in V/W).

   (b) Show that if $\| \cdot \|$ is a norm on V and W is a closed subspace, then

$$\|x + W\|_W := \inf_{w \in W} \|x + w\|$$

   is a norm on V/W.

(c) Show that, with the hypotheses in part b, $\pi : V \to V/W$, defined by $\pi(v) = v+W$, is a BLT.

(d) Show that if $(V, \|\cdot\|)$ is a Banach space and W is a closed subspace, then $(V/W, \|\cdot\|_W)$ is a Banach space. Hint: use part c and Folland Theorem 5.1.

(e) Show that if V is a Hilbert space and W is a closed subspace, then V/W is isometrically isomorphic to $W^\perp$.

Solution:

a. If $v_1' = v_1 + w_1, v_2' = v_2 + w_2$, with $w_1, w_2 \in W$, then

$$(v_1' + W) + (v_2' + W) = ((v_1 + w_1) + W) + ((v_2 + w_2) + W) = (v_1 + W) + (v_2 + W)$$

and so addition is well-defined. Similarly, scalar multiplication is well-defined.

The zero element is the equivalence class W and the additive inverse of $v + W$ is $-v + W$.

b. Positivity: Clearly $\|\cdot\|_W \geq 0$. If $\|x + W\|_W = 0$, then there exist $w_n \in W$ s.t. $\|x + w_n\| \to 0$ and so $w_n \to -x$. Since W is closed, $x \in W$ and so $x + W = W$, the zero element of V/W.

Homogeneity: If $\lambda \neq 0$,

$$\|\lambda x + W\|_W = |\lambda| \; \|x + \lambda^{-1}W\|_W = |\lambda| \; \|x + W\|_W$$

If $\lambda = 0$, then

$$\|\lambda x + W\|_W = \|W\|_W = 0.$$

Triangle inequality: Let $y, z \in X$. Given $\epsilon > 0$, let $u, v \in W$ s.t.

$$\|y + u\| \leq \|y + W\|_W + \epsilon, \;\; |z + v\| \leq \|z + W\|_W + \epsilon$$

The

$$\|(y + z) + W\|_W \leq \|y + z + u + v\| \leq \|y + u\| + \|z + v\| \leq \|y + W\|_W + \|z + W\|_W + 2\epsilon$$

Thus,

$$\|(y + z) + W\|_W \leq \|y + W\|_W + \|z + W\|_W$$

c. π is linear by the definitions of vector addition and scalar multiplication in V/W.

And $\|\pi(x)\| = \|x + W\| \leq \|x\|$ and so π is a BLT with $\|\pi\| \leq 1$ (it turns out that $\|\pi\| = 1$ but we won't need this).

d. Let $x_n + W$ be an absolutely convergent series in V/W, i.e. $\sum_n \|x_n + W\| < \infty$.

For each n, there exist $x_n' \in x_n+W$ s.t. $\|x_n'\| \leq \|x_n+W\|+1/n^2$. Thus, x_n' is absolute;y convergent in V. Since V is a Banach space, $\sum x_n'$ converges to some $x \in V$. But since π is a BLT it is continuuous and so

$$\pi(x) = \sum_n \pi(x_n') = \sum_n (x_n + W)$$

Thus, the series converges.

e. Define $\Phi : W^\perp \to V/W$ by $\Phi(z) = z + W$.

Linear:

$$\Phi(a_1 z_1 + a_2 z_2) = a_1 z_1 + a_2 z_2 + W = a_1(z_1 + W) + a_2(z_2 + W)$$

Norm-preserving: for $z \in W^\perp$ and $w \in W$, since $\langle w, z\rangle = 0$,

$$\|w + z\|^2 = \|w\|^2 + \|z\|^2 \geq \|z\|^2$$

and so

$$\|\Phi(z)\| = \inf_{w \in W} \|w + z\| = \|z\|$$

Injective: follows from norm-preserving.

Surjective: Given $x + W$, write $x = w + z$ with $w \in W, z \in W^\perp$. Then $x + W = z + W$ and so $\Phi(z) = x + W$.

5. (a) Let $X = \ell_\infty$ and $W = c_0$, a closed subspace of X. Let $x = (1, 1, 1, \cdots)$. Find $\inf_{y \in W} \|x - y\|_{\sup}$ and show that the inf is achieved but not uniquely.

   (b) Let

$$W = \{f \in C([0,1]) : \int_0^{1/2} f(t)dt - \int_{1/2}^1 f(t)dt = 1\}$$

   Find $\inf_{f \in W} \|f\|_{\sup}$ and show that the inf is not achieved at all.

Solution:

a. For all $y \in c_0$,

$$\lim_{n\to\infty} |x_n - y_n| = 1.$$

Thus, for all $y \in c_0$, $\|x - y\|_{\sup} \geq 1$. For any standard basis vector e_n, $\|x - e_n\|_{\sup} = 1$.

Thus, $\inf_{y \in W} \|x - y\|_{\sup} = 1$ and is achieved by any e_n.

So, in a Banach space (in contrast to a Hilbert space), the distince from a point to a closed subspace need not be achieved uniquely.

b. For all $f \in C([0,1])$,

$$\int_0^{1/2} f(t)dt \leq (1/2)\|f\|$$

and

$$-\int_{1/2}^1 f(t)dt \leq (1/2)\|f\|$$

So, if $f \in W$, then

$$1 = \int_0^{1/2} f(t)dt - \int_{1/2}^1 f(t)dt \leq (1/2)\|f\| + (1/2)\|f\| = \|f\|$$

and so $\|f\| \geq 1$ and so $\inf_{f \in W} \|f\| \geq 1$.

We claim that there does not exist $f \in W$ s.t. $\|f\| = 1$: If such an f, exists then by the above,

$$\int_0^{1/2} f(t)dt = (1/2), \quad - \int_{1/2}^{1} f(t)dt = (1/2)$$

and since f is continuous, $f(x) = 1$ for $0 \le x \le 1/2$ and $f(x) = -1$ for $1/2 \le x \le 1$, a contradiction.

We will construct $f_\epsilon \in W$ s.t. $\|f_\epsilon\| = 1 + \epsilon$. It follows that $\inf_{f \in W} \|f\|_{\sup} = 1$ and the inf is not achieved.

Given $\epsilon > 0$, let f_ϵ be the function whose graph is the polygonal path that connects $(0, 1+\epsilon)$ to $(1/2 - \epsilon/(1+\epsilon), 1+\epsilon)$ to $(1/2 + \epsilon/(1+\epsilon), -(1+\epsilon))$ to $(0, -(1+\epsilon))$.

Then $f_\epsilon \in W$ and $\|f_\epsilon\| = 1 + \epsilon$.

So, in a Banach space (in contrast to a Hilbert space), the distance from a point to a closed convex subset need not be achieved at all; one can re-cast this example to show that the distance from a point to a closed subspace need not be achieved at all .

Definitions for Problems 6 and 7:

A *topological space* $(X, \mathcal{T})$ is a set X together with a topology $\mathcal{T}$ which is a collection of subsets of X that is closed under arbitrary unions and finite intersections and includes the empty set and X.

An *open set* in a topological space $(X, \mathcal{T})$ is an element of $\mathcal{T}$. A *closed set* is the complement of an open set.

A subset K of a topological space is *compact* if every open cover of K has a finite subcover.

A mapping f from one topological space X to another Y is *continuous* if for every open set U in Y, $f^{-1}(U)$ is open in X.

A mapping f from one topological space X to another Y is an *open mapping* if for every open set U in X, $f(U)$ is open in Y.

A mapping f from one topological space X to another Y is a *closed mapping* if for every closed set U in X, $f(U)$ is closed in Y.

A topological space $(X, \mathcal{T})$ is *Hausdorff* if for all $x, y \in X, x \neq y$, there exist disjoint open sets U and V s.t. $x \in U$ and $y \in V$.

6. (a) Show that any metric space is Hausdorff.
   (b) Show that in a topological space a closed subset of a compact set is compact.
   (c) Show that in a Hausdorff space, any compact set is closed.
   (d) Let X and Y be topological spaces. Let $f : X \to Y$ be a continuous mapping and $K \subseteq X$ a compact set. Show that the image, $f(K)$, is compact.

(e) Show that any continuous map from a compact set to a Hausdorff space is a closed mapping.

(f) Show that a continuous map from a compact set to a Hausdorff space need not be an open mapping.

(g) Show that a bijective continuous map from a compact set in a topological space to a Hausdorff space is a homeomorphism.

Solution:

a. Given $x \neq y$ in a metric space, $B_{d(x,y)/2}(x)$ and $B_{d(x,y)/2}(y)$ are disjoint open balls and therefore disjoint open sets with $x \in B_{d(x,y)/2}(x)$ and $y \in B_{d(x,y)/2}(x)$.

b. Let L be a closed subset of a compact set K in a topological space X. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of L. Then $\{\{U_\alpha\}_{\alpha \in A}, L^c\}$ is an open cover of K. Since K is compact, the cover has a finite subcover $\{V_i\}_{i=1,\ldots,n}$ of K. This is also a finite subcover of L.

c. Let K be a compact subset of a topological space X. To show that K is closed, we show that K^c is open. For this, we show that for all $x \in K^c$, there is an open set U s.t. $x \in U \subset K^c$.

For each $y \in K$, there exist U_y, V_y be disjoint open sets s.t. $x \in U_y, y \in V_y$. Then $\{V_y\}_{y \in K}$ is an open cover of K, which must have a finite subcover $V_{y_1}, \ldots, V_{y_n}$. But then $x \in U := \cap_{i=1}^n U_{y_i}$ which is an open set disjoint from $\cup_{i=1}^n V_{y_i} \supset K$. Thus, U is an open set s.t. $x \in U \subset K^c$.

d. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of $f(K)$. Then $\{f^{-1}(U_\alpha)\}_{\alpha \in A}$ is an open cover of K. Since K is compact, there is a finite subcover $\{f^{-1}(U_i)\}_{i=1,\ldots,n}$ of K. But then $\{U_i\}_{i=1,\ldots,n}$ is a finite subcover of $f(K)$.

e. Let L be a closed subset of the domain, which is compact. By part b, L is compact and thus by part d, $f(L)$ is compact and thus by part c, $f(L)$ is closed.

f. The map from $[0,1]$ to the unit circle given by $f(\theta) = e^{2\pi i\theta}$ is continuous but not open because the image of the open set $[0, 1/2)$ is not open in the unit circle (here, we are using the relative topologies on the unit interval and the unit circle).

g. By part e, f is a closed mapping. Since f is bijective, f is an open mapping. Thus f^{-1} is continuous and so f is a homeomorphism.

7. Let $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha \in A}$ be a collection of topological spaces. Let

$$X = \Pi_\alpha X_\alpha = \{x = \{x_\alpha\}_\alpha : x_\alpha \in X_\alpha\}.$$

For a collection of subsets U_α in X_α, the corresponding *product subset* is

$$\Pi_\alpha U_\alpha = \{x \in X : x_\alpha \in U_\alpha \text{ for all } \alpha\}.$$

Let $\mathcal{E}$ denote the collection of all product subsets such that each U_α is open in X_α and for all but finitely many α, $U_\alpha = X_\alpha$.

The *product topology* $\mathcal{T}$ is the collection of all arbitrary unions of elements of $\mathcal{E}$.

The *projection map* $\pi_\alpha : X \to X_\alpha$ is defined by $\pi_\alpha(x) = x_\alpha$.

(a) Show that the product topology is indeed a topology.

(b) Show that the product topology is the smallest topology on X w.r.t which each projection map is continuous.

(c) Show that all projection maps are open mappings w.r.t. the product topology.

(d) Show that the topological space $(\mathbb{R}^n, \mathcal{T})$ where $\mathcal{T}$ is the collection of open sets in the Euclidean metric, is the product of n copies of the real line, each with the topology of open sets in the Euclidean metric on $\mathbb{R}$.

(e) Show that a projection map need not be a closed mapping.

Solution:

a. This follows from the fact that $\mathcal{E}$ is closed under finite intersections.

b. For any α and any subset $U_\alpha \subseteq X_\alpha$,

$$\pi_\alpha^{-1}(U_\alpha) = U_\alpha \times \prod_{\beta \neq \alpha} X_\beta$$

Thus, each π_α is continuous w.r.t. a topology $\mathcal{T}$ iff for every open set U_α in X_α, $U_\alpha \times \prod_{\beta \neq \alpha} X_\beta \in \mathcal{T}$. But since a topology is closed under finite intersections and abritrary unions, this is equivalent to the condition that $\mathcal{T}$ contains the product topology.

So, the product topology is the smallest topology on X w.r.t which each projection map is continuous.

c. Clearly π_α maps every element of $\mathcal{E}$ to an open set in X_α. Thus, it maps every union of elements of $\mathcal{E}$ to an open set in X_α. But this means that all open sets in the product topology are mapped to open sets in X_α.

d. The product topology $\mathcal{T}_1$ on $\mathbb{R}^n$ is the collection of arbitrary unions of sets of the form

$$U_1 \times \cdots \times U_n$$

where each U_i is an open set in $\mathbb{R}$. Since each open set in $\mathbb{R}$ can be written as a union of open intervals, $\mathcal{T}_1$ is the collection of arbitrary unions of sets of the form

$$(a_1, b_1) \times \cdots \times (a_n, b_n) \tag{3}$$

The Euclidean topology $\mathcal{T}_2$ on $\mathbb{R}^n$ is the collection of open balls.

Since each open ball can be written as a union of sets of the form (3) and each set of the form (3) can be written as a union of open balls, the topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ coinccide.

e. The set $\{(x, y) : y = 1/x,\ x > 0\}$ is closed in $\mathbb{R}^2$ but its image via the projection map, $\pi_1 : \mathbb{R}^2 \to \mathbb{R}, \pi_1((x, y)) = x$ is $(0, \infty)$ which is not a closed set in the topology of $\mathbb{R}$.