HW4, Due Friday, March 8, 11AM

1. (a) Show that the kernel of a BLF on a Banach space is closed.
   (b) Shat that the orthogonal complement of the kernel of a BLF on Hilbert space has dimension at most one.

2. Show that if $\|\cdot\|$ is a norm on a real vector space that satisfies the parallelogram law then
$$\langle x, y\rangle := (1/4)(\|x+y\|^2 - \|x-y\|^2) \tag{1}$$
is an inner product whose induced norm agrees with $\|\cdot\|$. And thus show that a real Banach space is a real Hilbert space iff its norm satisfies the parallelogram law.

   Hint: To prove bilinearity of $\langle x, y\rangle$,

   (a) first show
   $$\langle v+w, u\rangle = \langle v, u\rangle + \langle w, u\rangle$$
   (b) then use part (a) to show for all rational α
   $$\langle \alpha x, y\rangle = \alpha\langle x, y\rangle \tag{2}$$
   (c) then show (2) for all α.

3. Recall that for a complex Hilbert space H and $y \in H$, $y^*(x) = \langle x, y\rangle$ is a BLF.

   Show that $\langle y^*, z^*\rangle_{H^*} := \langle z, y\rangle_H$ is an inner product on H^*.

4. Let V be a vector space and W be a subspace of V. Let V/W denote the set of equivalence classes of the relation $v_1 \sim v_2$ iff $v_1 - v_2 \in W$. So, the equivalence classes are of the form $[v] = v + W$. Define a vector space structure on V/W by
$$(v_1 + W) + (v_1 + W) = (v_1 + v_2) + W$$
and
$$\lambda(v + W) = \lambda v + W.$$
   (a) Show that V/W is a well-defined vector space (just show that the vector addition and scalar multiplication operations on V/W are well-defined, identify the zero element of V/W amd the additive inverses in V/W).
   (b) Show that if $\|\cdot\|$ is a norm on V and W is a closed subspace, then
   $$\|x + W\|_W := \inf_{w\in W} \|x + w\|$$
   is a norm on V/W.
   (c) Show that, with the hypotheses in part b, $\pi : V \to V/W$, defined by $\pi(v) = v+W$, is a BLT.

(d) Show that if $(V, \|\cdot\|)$ is a Banach space and W is a closed subspace, then $(V/W, \|\cdot\|_W)$ is a Banach space. Hint: use part c and Folland Theorem 5.1.

(e) Show that if V is a Hilbert space and W is a closed subspace, then V/W is isometrically isomorphic to $W^\perp$.

5. (a) Let $X = \ell_\infty$ and $W = c_0$, a closed subspace of X. Let $x = (1, 1, 1, \cdots)$. Find $\inf_{y \in W} \|x - y\|_{\sup}$ and show that the inf is achieved but not uniquely.

(b) Let

$$W = \{f \in C([0,1]) : \int_0^{1/2} f(t)dt - \int_{1/2}^1 f(t)dt = 1\}$$

Find $\inf_{f \in W} \|f\|_{\sup}$ and show that the inf is not achieved at all.

Definitions for Problems 6 and 7:

A *topological space* $(X, \mathcal{T})$ is a set X together with a topology $\mathcal{T}$ which is a collection of subsets of X that is closed under arbitrary unions and finite intersections and includes the empty set and X.

The *relative topology* on a subset A of a topological space $(X, \mathcal{T})$ is $\{A \cap U : U \in \mathcal{T}\}$

An *open set* in a topological space $(X, \mathcal{T})$ is an element of $\mathcal{T}$. A *closed set* is the complement of an open set.

A topological space $(X, \mathcal{T})$ is *compact* if every open cover of X has a finite subcover.

A subset of a topological space is compact if it is compact in its relative topology.

A mapping f from one topological space X to another Y is *continuous* if for every open set U in Y, $f^{-1}(U)$ is open in X.

A mapping f from one topological space X to another Y is an *open mapping* if for every open set U in X, $f(U)$ is open in Y.

A mapping f from one topological space X to another Y is a *closed mapping* if for every closed set U in X, $f(U)$ is closed in Y.

A topological space $(X, \mathcal{T})$ is *Hausdorff* if for all $x, y \in X, x \neq y$, there exist disjoint open sets U and V s.t. $x \in U$ and $y \in V$.

6. (a) Show that any metric space is Hausdorff.

(b) Show that in a topological space a closed subset of a compact set is compact.

(c) Show that in a Hausdorff space, any compact set is closed.

(d) Let X and Y be topological spaces with X compact. Let $f : X \to Y$ be a continuous mapping Show that the image, $f(X)$, is compact.

(e) Show that any continuous map from a compact space to a Hausdorff space is a closed mapping.

(f) Show that a continuous map from a compact space to a Hausdorff space need not be an open mapping.

(g) Show that a bijective continuous map from a compact space to a Hausdorff space is a homeomorphism.

7. Let $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha \in A}$ be a collection of topological spaces. Let

$$X = \Pi_\alpha X_\alpha = \{x = \{x_\alpha\}_\alpha : x_\alpha \in X_\alpha\}.$$

For a collection of subsets U_α in X_α, the corresponding *product subset* is

$$\Pi_\alpha U_\alpha = \{x \in X : x_\alpha \in U_\alpha \text{ for all } \alpha\}.$$

Let $\mathcal{E}$ denote the collection of all product subsets such that each U_α is open in X_α and for all but finitely many α, $U_\alpha = X_\alpha$.

The *product topology* $\mathcal{T}$ is the collection of all arbitrary unions of elements of $\mathcal{E}$.

The *projection map* $\pi_\alpha : X \to X_\alpha$ is defined by $\pi_\alpha(x) = x_\alpha$.

(a) Show that the product topology is indeed a topology.

(b) Show that the product topology is the smallest topology on X w.r.t which each projection map is continuous.

(c) Show that all projection maps are open mappings w.r.t. the product topology.

(d) Show that the topological space $(\mathbb{R}^n, \mathcal{T})$ where $\mathcal{T}$ is the collection of open sets in the Euclidean metric, is the product of n copies of the real line, each with the topology of open sets in the Euclidean metric on $\mathbb{R}$.

(e) Show that a projection map need not be a closed mapping.